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Solving the pentahedron problem☆

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HIGHLIGHTS

- Reduction of the pentahedron problem to a well-constrained system of 3 equations in 3 unknowns.
- A considerable performance enhancement (×42) over classical formulation.
- Existence of 3D parallel solutions for generic 3D pentahedron problems is shown.
- Interesting properties of the solution set are studied.
- Discussion of how the pentahedron interesting properties generalize for other polyhedra.

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ABSTRACT

Nowadays, all geometric modelers provide some tools for specifying geometric constraints. The 3D pentahedron problem is an example of a 3D Geometric Constraint Solving Problem (GCSP), composed of six vertices, nine edges, five faces (two triangles and three quadrilaterals), and defined by the lengths of its edges and the planarity of its quadrilateral faces. This problem seems to be the simplest non-trivial problem, as the methods used to solve the Stewart platform or octahedron problem fail to solve it. The naive algebraic formulation of the pentahedron yields an under-constrained system of twelve equations in eighteen unknowns. Even if the use of placement rules transforms the pentahedron into a well-constrained problem of twelve equations in twelve unknowns, the resulting system is still hard to solve for interval solvers. In this work, we focus on solving the pentahedron problem in a more efficient and robust way, by reducing it to a well-constrained system of three equations in three unknowns, which can be solved by any interval solver, avoiding by the way the use of placement rules since the new formulation is already well-constrained. Several experiments showing a considerable performance enhancement (×42) are reported in this paper to consolidate our theoretical findings. Throughout this paper, we also emphasize some interesting properties of the solution set, by showing that for a generic set of parameters, solutions in the form of 3D parallel edge pentahedra do exist almost all the time, and by providing a geometric construction for these solutions. The pentahedron problem also admits degenerate 2D solutions in finite number. This work also studies how these interesting properties generalize for other polyhedra.

1. Introduction

Geometric Constraint Solving Problems (GCSPs) have retained much of the researchers attention since several decades [1–7]. This attention may be justified by the advances in computing systems, in terms of both hardware capabilities and software facilities, which translated into a growing need for new CAD/CAM techniques and opened new perspectives for the implementation of researchers ideas. Although there exist a large number of GCSP-related works, expressing and solving geometric constraint systems is still an active research topic and much more effort has to be done in this direction.

This paper considers a particular GCSP problem: the 3D pentahedron. A pentahedron is a polyhedron in 3D, not necessarily convex, consisting of five faces (Fig. 1). Two faces are triangles, while

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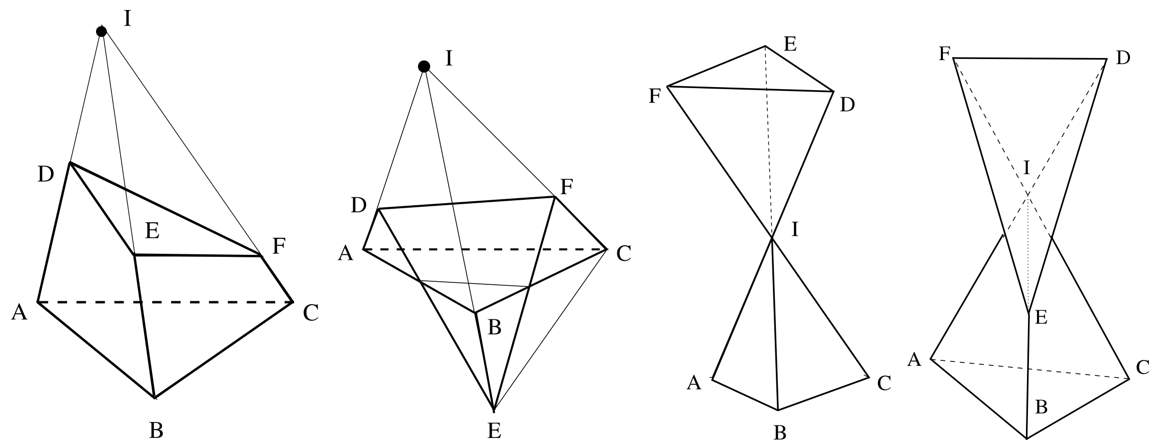

Fig. 1. A zoo of 3D pentahedra. All lines *AD*, *BE*, and *CF* which join the two triangles *ABC* and *DEF* concur at point *I*, according to Desargues' theorem.

the remaining three faces are planar quadrilaterals. The lengths of the nine edges are given in the problem formulation. Thus, the problem has twelve constraints: nine lengths of the nine edges, plus three coplanarity conditions of the three quadrilateral faces. Under mild assumptions, this problem is well-constrained modulo an isometry (rigid body motion and symmetry). The pentahedron may be concave and self-intersecting. As an example, two of its triangular faces may intersect each other (Fig. 1).

Why is the pentahedron problem so interesting? First, its statement is simple, but its resolution is difficult. We cannot solve it with computer algebra, even when the distance parameters are instantiated with numeric values. We cannot solve it with Cayley–Menger determinants or their generalizations [8–10], though the pentahedron problem seems very close to the octahedron problem (also named the Stewart platform) as both polyhedra have six vertices, and Cayley–Menger determinants give an elegant solution to the octahedron problem.

The pentahedron problem is even unsolvable with the reparameterization method proposed by Gao et al. [11] to solve many 3D simple problems similar to the pentahedron problem. In their work, Gao et al. detected or introduced one key unknown u (an angle or a length) and expressed all the other unknowns of the problem as a function of u. By construction, all the constraints of the problem, except one constraint called the ignored constraint, are satisfied. Therefore, it remains to only vary the key unknown u, i.e., to plot or sample a curve parameterized by u, and to detect when the ignored constraint is satisfied. Unfortunately, despite the apparent simplicity of the pentahedron, two key unknowns are needed to solve this problem.

The first reason behind our focus on the study of the pentahedron problem in this paper is that the latter has not yet been solved. To the best of our knowledge, no work in the literature has focused on this problem, though a recent Ph.D. thesis made an allusion to it [12]. The main contribution of our paper is the first method of resolution of the pentahedron which exploits a geometric property of the 3D pentahedron. This permits to drastically reduce the number of unknowns and equations of the problem. At the end, we obtain a system of only three equations in three unknowns, that is much smaller than the naive version of twelve equations in twelve unknowns, and which is easily solvable with either a wide range of interval solvers, or with homotopic solvers. Our experiments showed that the interval analysis library *ALIAS-C++* [13] achieved a performance gain of $\times 42$ for the reduced pentahedron formulation, compared to the naive one.

A second reason making the pentahedron problem very interesting is the structure of its solutions, which is unexpectedly rich. Up to an isometry, this problem admits three finite sets of solutions: (1) a first set of 3D generic solutions, (2) a finite set of 3D special solutions where three edges are parallel, and (3) a set of 2D solutions.

A third reason motivating our work is the opportunity given by this problem for studying some interesting issues on a simple but non-trivial problem. We only mention the following important issue: what are the rigidity or the flexibility conditions for the pentahedron problem? This issue is essential in many domains such as robotics and constitutes a challenge for computer algebra. Remember that the pentahedron, or any constrained object, is flexible when it can be continuously deformed, while fulfilling at the same time all the problem constraints during its deformation. For instance, it can be easily seen that when the two triangular faces of the pentahedron have equal lengths a, b, and c, and the three edges linking these two faces have the same length h, i.e., the three quadrilateral faces are parallelepipeds, then the pentahedron is flexible. More precisely, it has three degrees of freedom. For all possible directions of a 3D line, there exists a pentahedron solution, such that the equal-length edges of this pentahedron are oriented the same as that 3D line. It is easy to compute the degree of flexibility of a given pentahedron by simply computing the rank and co-rank of its Jacobian. However, computing the general algebraic conditions for flexibility or rigidity is a hard problem.

The rest of this paper is structured as follows: Section 2 describes the classical formulation of the pentahedron problem, which results in a system of twelve equations in twelve unknowns. Section 3 presents our new and reduced formulation of the pentahedron problem that yields a systems of three equations in three unknowns, and compares the performance of the interval solver *ALIAS-C++* on the two formulations. In Section 4, we show that for a generic set of length parameters, the corresponding pentahedron problem admits almost all the time solutions in the form of 3D pentahedra with parallel edges, and we provide a geometric construction for them. In Section 5, we consider planar solutions to the pentahedron problem. In Section 6, we discuss the degenerate case of 3D simple pentahedra and provide a solution for this configuration. Finally, in Section 7, we study how the interesting properties of the pentahedron problem and the proposed resolution method can extend or not to other simple polyhedra.

2. The classical 3D pentahedron GCSP

A GCSP is composed of a set of geometric objects, whose placement must fulfill a set of geometric constraints. The 3D pentahedron problem is composed of six points: A, B, C, D, E, F. Triples of points ABC and DEF constitute the vertices of the two triangular faces of the pentahedron, while the remaining three quadrilateral faces denoted as F_1, F_2, and F_3 have respective vertices $ABED$, $BCFE$, $CADF$. See Fig. 2(a).