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Topologically guaranteed bivariate solutions of under-constrained multivariate piecewise polynomial systems ☆

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HIGHLIGHTS

- A subdivision algorithm for 2-DOF nonlinear algebraic systems.
- Topologically guaranteed subdivision termination criteria in $\mathbb{R}^n$, $n \geq 3$.
- A tessellation method for two-manifolds in $\mathbb{R}^n$.

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ABSTRACT

We present a subdivision based algorithm to compute the solution of an under-constrained piecewise polynomial system of $n-2$ equations with n unknowns, exploiting properties of B-spline basis functions. The solution of such systems is, typically, a two-manifold in $\mathbb{R}^n$. To guarantee the topology of the approximated solution in each sub-domain, we provide subdivision termination criteria, based on the (known) topology of the univariate solution on the domain's boundary, and the existence of a one-to-one projection of the unknown solution on a two dimensional plane, in $\mathbb{R}^n$. We assume the equation solving problem is regular, while sub-domains containing points that violate the regularity assumption are detected, bounded, and returned as singular locations of small (subdivision tolerance) size. This work extends (and makes extensive use of) topological guarantee results for systems with zero and one dimensional solution sets. Test results in $\mathbb{R}^3$ and $\mathbb{R}^4$ are also demonstrated, using error-bounded piecewise linear approximations of the two-manifolds.

1. Introduction and related work

The general problem discussed in this paper is the solution of equation systems of the form:

$$F(\bar{x}) = \bar{0}, \tag{1}$$

where the multivariate function F is a piecewise polynomial, defined on some compact n-dimensional box $D \subset \mathbb{R}^n$ with values in $\mathbb{R}^{n-2}$. We assume $\bar{0} \in \mathbb{R}^{n-2}$ is a regular value (as defined in Section 2) of F.

The need for an efficient and robust method for finding all solutions of equation systems (1) in a given domain arises in a variety of fields such as Computer Aided Design (CAD), engineering, robotics, and in fact whenever the geometric constraints of the problem can be formulated as a set of algebraic (non-linear in general) equations. Bisector surfaces [1], sweep-surfaces [2], medical iso-surfaces [3], and the possible states of any 2-DOF (Degrees of Freedom) kinematic system (Ch. 06 of [4]) are all examples of application domains that can be mapped to an under-constrained system with a bivariate solution in $\mathbb{R}^n$, where $n \geq 3$.

The general problem of solving non-linear algebraic constraints with any solution set's dimension (not necessarily two) is typically addressed via either *local* methods and/or *global* methods. The local methods refer to a family of numeric iterative algorithms, such as the Newton–Raphson procedure or other prediction–correction methods. Although very useful and typically of quadratic convergence, these techniques depend on the quality of the initial candidate/starting point, and cannot guarantee the global solution: all roots or all connected components of the solution manifold, and its topological properties.

As for global methods, the problem of finding all (real or complex) solutions for equations such as (1) has been mainly addressed via three major approaches. *Algebraic geometry techniques* (elimination theory and the use of Gröbner bases, [4]) and *Homotopy*

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techniques [5] are typically less efficient for finding only the real roots in a bounded domain, due to various practical considerations (for a detailed survey refer to [6,7]). The third class of methods (and the one discussed in this paper), are the *subdivision techniques*, which are reviewed in Section 3, and are the framework for the results introduced herein.

The main result in this paper is a topologically guaranteed subdivision termination condition for equation systems with bivariate solutions (two-manifolds). The technique extends and exploits previously achieved topological guarantee results for the zero and one dimensional solution manifolds in [8–10]. Since these results play a significant role in the development of the methods introduced in this paper, they are reviewed in more detail in Section 3. Subdivision algorithms with topological guarantee are known for implicit regular surfaces in $\mathbb{R}^3$, for example as in [11,12]. To the best of our knowledge, the criteria introduced in this paper is the first to provide topological guarantee for two-manifold solutions in $\mathbb{R}^n$, when the co-dimension is greater than one ($n > 3$).

The problem of guaranteeing the topological properties of manifolds given as (solutions of) algebraic constraints has been widely addressed. Methods for guaranteeing the topology of implicit plane curves are presented in [13,14], and are based on locating the critical points and subdividing the domain such that each piece of the solution is a monotone arc. Further, a numerical method with topological guarantee for implicit planar curves is given in [15], which also detects isolated singularities and computes their degree, using the number of connected components of certain topological structure in the neighborhood of the singularity. The topology of implicit surfaces has also been extensively investigated in $\mathbb{R}^3$, within the framework of ϵ-sampling ([16] and more), in the contexts of polygonization and rendering [3,17–19], with results for the non-regular cases as well [20,21]. More recent results for level sets of a given implicit function in $\mathbb{R}^3$ are in [11,12], presenting algorithms for correct connectivity of the reconstructed surface by a careful case analysis and subdivision until ambiguity can be resolved, using additional information such as parameterizability and bounding gradient norm. In [22], methods for contouring one- and two-manifolds, generally in $\mathbb{R}^n$, are proposed, but without a topological guarantee. In [21], results from Morse theory are used to locate and classify singular points of an implicit surface in $\mathbb{R}^3$, and conclude correct topology via interactive polygonization. Other related aspects of nonlinear systems are treated in [23] on dimension reducing, in [24] on expression trees representation, in [25] on parallel computation and convergence guarantee of Newton–Raphson via the Kantorovich theorem, and more. Our focus in this paper is mainly on the subdivision stage, the topological properties of the solution set (rather than the numeric properties) in an attempt to guarantee the topology of the result, and we make frequent use of the properties of the B-spline representation.

The rest of this paper is organized as follows: Section 2 introduces the terminology and the problem setting. Section 3 provides a brief review of the subdivision paradigm for algebraic constraints solving, focusing on useful results to our problem. Section 4 provides the solution to the problem of a topologically guaranteed subdivision termination criteria for under-determined systems with bivariate solution spaces. Section 5 introduces a triangulation method for the numeric reconstruction in the topologically guaranteed sub-domains. Some examples are given in Section 6, and finally, Section 7 concludes and discusses further optional problems for this research.

2. Notation and terminology

We now provide the problem formulation, state general assumptions, and review frequently used basic concepts. Let F be a multivariate function defined on some compact, n-dimensional box: $D = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$, with values in $\mathbb{R}^{n-2}$. For differentiability considerations, F can always be viewed as a function defined on some open set $U \subset \mathbb{R}^n$ containing D. Unless otherwise stated, F is at least C^1 smooth, and is typically given in a tensor product Bézier or B-spline form (a detailed study of these representations can be found in [26]), generally represented as:

$$F(x_1, \ldots, x_n) = \sum_{i_1} \cdots \sum_{i_n} P_{i_1 \ldots i_n} B_{i_1, m_1}(x_1) \ldots B_{i_n, m_n}(x_n). \tag{2}$$

The points $P_{i_1 \ldots i_n} \in \mathbb{R}^{n-2}$ are the *control points* of F, and the B_{i_j, m_j}'s are the B-spline basis functions of order m_j of F. The zero set of F is denoted by:

$$Z = \{\bar{x} \in U : F(\bar{x}) = \bar{0} \in \mathbb{R}^{n-2}\} = F^{-1}(\bar{0}).$$

Our main interest is in $Z \cap D$, the region of the zero set belonging to the compact domain. The concepts we now refer to, can be found in textbooks on topology and differentiability such as [27,28]. Recall that a *homeomorphism* is a continuous function between topological spaces that has a continuous inverse. Homeomorphisms are the equivalence relation between topological spaces. When we later claim that the "topology of the solution set is guaranteed", we mean that we have successfully classified the solution set "up to homeomorphism".

Omitting the precise and somewhat technical definition [27,29], recall that a two dimensional sub-manifold of $\mathbb{R}^n$, ($n \geq 2$) is a subset $M \subset \mathbb{R}^n$, such that every $p \in M$ has a neighborhood (in the topology inherited from $\mathbb{R}^n$) that is homeomorphic to (an open subset of) the plane $\mathbb{R}^2$. The ability to treat the solution set as a manifold is made possible by the regularity assumption which we now describe. Denote by f_i the i'th scalar component of F. The *Differential of F at a point* $p \in D$ is the linear map denoted by dF_p, represented in the standard bases by the $(n-2) \times n$ matrix:

$$[dF_p]_{ij} = \frac{\partial f_i}{\partial x_j}(p); \quad i = 1, \ldots, n-2;\ j = 1, \ldots, n.$$

Definition 1. Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^{n-2}$ be a differentiable mapping of an open set $U \subset \mathbb{R}^n$. A point $p \in U$ is defined to be a *critical point* of F if the differential $dF_p : \mathbb{R}^n \to \mathbb{R}^{n-2}$ is not surjective. The image $F(p)$ of a critical point is called a *critical value* of F. A point $a \in \mathbb{R}^{n-2}$ that is not a critical value is called a *regular value* of F.

From Definition 1, it is evident that for $c \in \mathbb{R}^{n-2}$ to be a regular value of F, dF_p is surjective *for all* $p \in F^{-1}(c)$. When this is the situation, $F^{-1}(c)$ is called a *regular level set*. Our regularity assumption is that the solution $F^{-1}(\bar{0})$ is a regular level set. As another interpretation, the regularity assumption is equivalent to the linear independence of the $n-2$ gradients of f_i, for all $p \in Z$.

The term "general position" usually refers to a situation where an arbitrarily small perturbation of non-regular input, re-attains regularity. Put differently: the configurations of the input that do not satisfy the assumption, occur with zero probability.[1] Indeed, this is precisely our case, as stated in Sard's theorem [27], which roughly says that the set of critical values of a smooth function has measure zero in the range. In our setting, this is equivalent to saying that if $Z = F^{-1}(\bar{0})$ is *not* a regular level set, then for any arbitrarily small $\epsilon > 0$, there is another value $c \in \mathbb{R}^{n-2}$ such that $F^{-1}(c)$ *is* a regular level set, and $\|c\| < \epsilon$. Although rare in the pure mathematical sense, a non-surjective differential is in fact a common configuration in real life problems (i.e. tangential curves/surfaces). We briefly address this issue in Section 7.

Finally, the following theorem is what enables us to use the properties of a manifold for regular level sets:

[1] The precise definition for general position varies according to context. A more detailed exposition can be found in Ch. 1 of [30].