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Constructing developable surfaces by wrapping cones and cylinders☆

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HIGHLIGHTS

- An effective technique for modeling the bending of paper using cones and cylinders.
- A methodology for producing complicated developable surfaces from a planar figure.
- Interactive control of paper bending from a user-specified displacement.

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ABSTRACT

We model developable surfaces by wrapping a planar figure around cones and cylinders. Complicated developables can be constructed by successive mappings using cones and cylinders of different sizes and shapes. We also propose an intuitive control mechanism, which allows a user to select an arbitrary point on the planar figure and move it to a new position. Numerical techniques are then used to find a cone or cylinder that produces the required mapping. Several examples demonstrate the effectiveness of our technique.

1. Introduction

A developable surface is a special type of ruled surfaces with zero Gaussian curvature which can be flattened on to a plane without distortion [1,2]. Examples include simple surfaces such as cones and cylinders, as well as tangent or rectifying developables derived from spatial curves. Due to their isometric property, developable surfaces have been widely used in applications such as architecture, garment design, sheet metal and origami [3–5].

Typical approaches include the representation of developable as ruled Bézier or B-spline surfaces with non-linear developability constraints [6–9]. A dual-space approach is mathematically elegant, but it requires significant geometric expertise for an intuitive shape control [2,6]. Many practical methods have been proposed for constructing developable surfaces from other representations, including point clouds [10,11], boundary curves [12,13] and polygonal meshes [4,5,14,15].

Recently, a lot of research [16–19] has been done on the modeling of paper-like surfaces and simulating their deformations. Kergosien et al. [16] simulated the bending and creasing of paper-like sheets under external and internal forces. Hong et al. [17] presented a method for simulating page turning for use in 3D electronic books (e-books). They use a cone of variable angle to simulate the bending of paper while maintaining its developability. However, their method only allows restricted deformations, and does not provide an interactive control mechanism. Bo and Wang [18] modeled paper bending as a rectifying developable defined by one of its geodesics. More recently, Solomon et al. [19] proposed a versatile method for constructing developable surfaces with folds, bends and curved folds from a planar fold configuration. For smoothly bent region, they subdivide the rulings and relax them by minimizing mean curvature bending energy.

In this paper, we also deal with the problem of paper bending, but take a different approach. We extend the technique of Hong et al. [17] to a general modeling tool for constructing developable surfaces. We place cones and cylinders on a planar figure with an arbitrary boundary shape, and wrap them in the planar figure to generate a developable surface. Complicated surfaces can easily be obtained by introducing a succession of cones and cylinders of different sizes and shapes. We also propose an intuitive control technique, in which a user selects an arbitrary point **p** on the planar figure and moves it to a new position **q**. Finding a cone or cylinder which maps **p** to **q** allows the subsequent development to appear as a direct manipulation. Selection of the cone or cylinder is formulated as a problem of root finding or optimization of non-linear functions, with only a few control parameters, which can be efficiently solved by a standard numerical technique.

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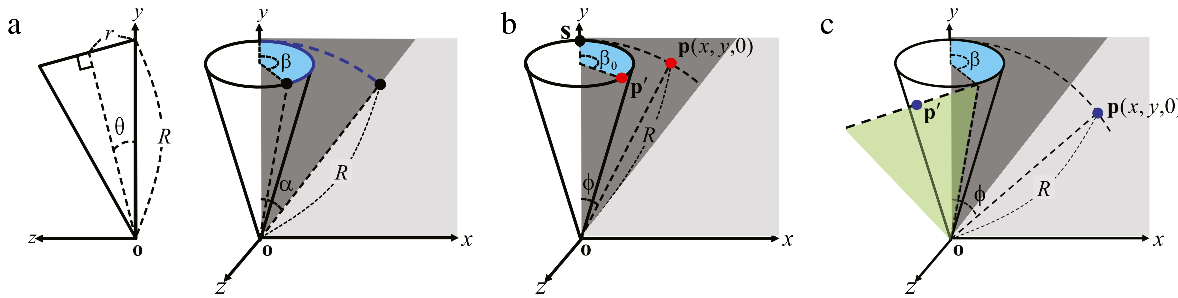

Fig. 1. Cone mapping: (a) bending region determined by the angle β, (b) points in the bending region, (c) points in the planar region.

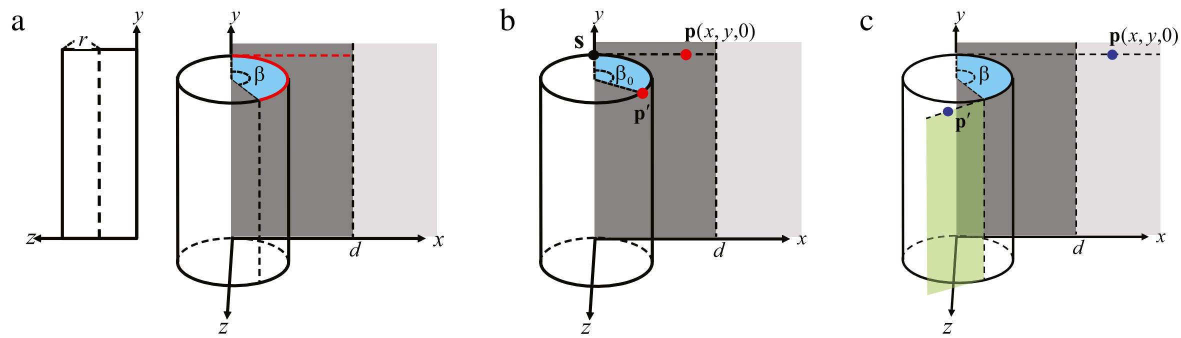

Fig. 2. Cylinder mapping: (a) bending region determined by the angle β, (b) points in the bending region, (c) points in the planar region.

The main contributions of this paper can be summarized as follows:

- A simple and effective technique for modeling the bending of a sheet of paper using cones and cylinders.
- A methodology for the successive application of cones and cylinders of different sizes and shapes to produce complicated developable surfaces from an arbitrary planar figure.
- Interactive control of bending by finding a cone or cylinder which corresponds to a specified displacement by the user.

2. Modeling paper bending

Cones and cylinders are well-known developable surfaces. By using them as primitives for modeling paper bending, we can guarantee that the deformed paper is a developable surface.

Cone mapping: We employ a circular cone $\mathcal{C}_O$ with half-angle θ. Its apex is located at the origin of a local coordinate system and one generatrix coincides with the y-axis of that system, as shown in Fig. 1(a). We are modeling a cone placed on paper, which wraps the cone to an angle β. The bending region (shown in dark gray in Fig. 1(a)) of the paper is determined by the angle α, which is equal to $\beta \sin\theta$ ($\because \alpha R = \beta r$).

All points $\mathbf{p} = (x, y, 0)$ in the bending region (see Fig. 1(b)) satisfy $\phi \leq \alpha$, where $\phi = \arctan(x/y)$ is the angle between the y-axis and $\overrightarrow{\mathbf{op}}$, and these points are mapped to the points $\mathbf{p}'$ on $\mathcal{C}_O$ as follows:

$$\mathbf{p}' = R_x(\theta)R_y(-\beta_0)R_x(-\theta)\mathbf{s},$$

where $\beta_0 = \phi/\sin\theta$, $\mathbf{s} = (0, \sqrt{(x^2+y^2)}, 0)$, and R_* represents a rotation about the corresponding axis. The points in the remaining planar region ($\phi > \alpha$, shown in Fig. 1(c)) can be transformed as follows:

$$\mathbf{p}' = R_x(\theta)R_y(-\beta)R_x(-\theta)R_z(\alpha)\mathbf{p}.$$

Cylinder mapping: We employ a circular cylinder $\mathcal{C}_Y$ with radius r, which is tangent to the xy-plane of a local coordinate system (see Fig. 2(a)). To model paper being wound around the cylinder by an angle β requires a bending region of length $d = r\beta$. In a similar way to the cone mapping, points $\mathbf{p} = (x, y, 0)$ in the bending region (see Fig. 2(b)) satisfy $x \leq d$, and are mapped to points $\mathbf{p}'$ on $\mathcal{C}_Y$ as follows:

$$\mathbf{p}' = T(0, 0, r)R_y(-\beta_0)T(0, 0, -r)\mathbf{s},$$

where $\beta_0 = x/r$, $\mathbf{s} = (0, y, 0)$, and $T(\cdot)$ represents a 3D translation. Points $\mathbf{p} = (x, y, 0)$ in the planar region ($x > d$, shown in Fig. 2(c)) can be transformed as follows:

$$\mathbf{p}' = T(0, 0, r)R_y(-\beta)T(-d, 0, -r)\mathbf{p}.$$

Generation of ruling lines: In general, developable surfaces can be represented efficiently by ruling line segments which are straight. Since the paper deformed by cones and cylinders is developable, we compute the ruling line segments and use them to construct a tessellation consisting of strips. On the bending region, ruling lines are obtained by finding the intersections of lines passing through the origin with the paper boundary (see Fig. 4(a)), and the end-points of the resulting line segments are mapped on to a cone or cylinder as appropriate. To preserve the sharp corner of bending region, we force some ruling lines to pass through corner-points (see blue one in Fig. 4(a)). Regions where the paper remains flat are modeled by their transformed corner-points (shown in red in Fig. 4(a)). Fig. 4(b) shows how a bent piece of paper is modeled by ruling line segments and transformed planar regions. Fig. 4(c) shows the result of mapping a tiled texture on to a model of bent paper and we see that no distortion is apparent.

Our technique provides a compact representation as well as computational efficiency, because only a few intersection points are involved in transformation and tessellation, and the number of ruling line segments can interactively be controlled by the user at runtime. It would be quite inefficient if tessellation preceded transformation, as shown in Fig. 4(d).

Fig. 3 shows how complicated deformations of paper can be modeled by successive mappings using several different cones and cylinders. Note that the planar parts are shown in dark gray and