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Contour curves and isophotes on rational ruled surfaces

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ABSTRACT

Ruled surfaces, i.e., surfaces generated by a one-parametric set of lines, are widely used in the field of applied geometry. An isophote on a surface is a curve consisting of those surface points whose normals form a constant angle with a fixed vector. Choosing the angle equal to $\pi/2$ we obtain a special instance of the isophote – the so called contour curve. While contours on rational ruled surfaces are rational curves, this is no longer true for the isophotes. Hence we will provide a formula for their genus. Moreover we will show that the only surfaces with a rational generic contour are just the rational ruled surfaces and a particular class of cubic surfaces. In addition we will deal with a reconstruction of ruled surfaces from their silhouettes.

1. Introduction

Contour curves and isophotes are characteristic curves on surfaces, which play a key role, for example in recognition and interrogation of surfaces. For a given surface the contour with respect to a viewpoint consists of the points of the surface where the tangent plane passes through the viewpoint. Projecting the contour into the plane of projection we obtain the so called silhouette. Some related studies on contours, silhouettes and their applications can be found e.g. in Seong et al. (2006); Bizzarri and Lávička (2013); Bizzarri et al. (2015). An isophote is defined as a locus of points whose surface normals enclose a constant angle with a fixed direction. Hence a contour can be understood as a special case of the isophote for a choice of the angle equal to $\pi/2$. See e.g. Poeschl (1984); Kim and Lee (2003); Aigner et al. (2009); Dogan and Yayli (2015) for papers on computation and properties of isophotes.

Nowadays, the NURBS representation (where NURBS stands for Non-Uniform Rational B-Spline) is considered as a universal standard in geometric modelling and computer-aided design. The curves/surfaces are built from glued pieces of curves/surfaces parametrized by rational functions. Given a rational parametrization, one can eliminate the parameters and obtain a collection of non-parametric polynomial equations defining the original object. Thus the curves and surfaces admitting a rational parametrization are algebraic varieties. However, the converse is not true any more. For example, to each curve we can assign a non-negative integer – its genus. The only curves possessing rational parametrizations are the curves with vanishing genus. Starting with a rational surface the contours and isophotes are algebraic curves, although they are not typically rational ones. This is a reason why we try to understand the classes of surfaces where these curves are manageable.

The remainder of this paper is organized as follows: After recalling the definitions of contour curves and isophotes, the second section is devoted to basic properties of rational ruled surfaces. In Section 3 we study algebro-geometric properties of

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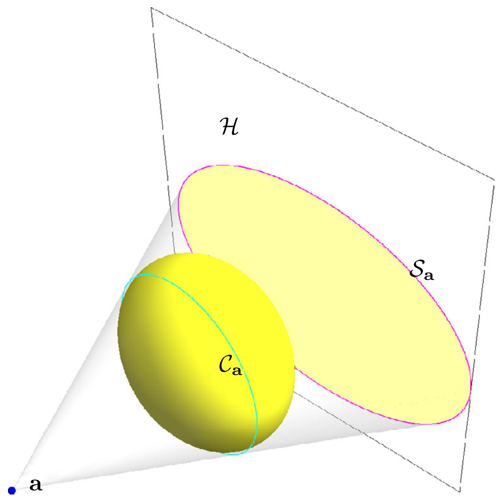

Fig. 1. The contour $\mathcal{C}_{\mathbf{a}}$ (cyan) w.r.t. the point **a** and the silhouette $\mathcal{S}_{\mathbf{a}}$ (magenta) as the projection of the contour into the plane $\mathcal{H}$.

the curves of interest. Since rational curves play a prominent role in computer-aided design, we will classify all the surfaces with rational generic contour curves. It turns out that the rational ruled surfaces are the most interesting case and thus the paper is mostly devoted to them. The situation with isophotes is even more complicated. In contrast to the standard definition, the algebraic setting forces to consider an isophote to be a locus of points whose normal enclose the angle ϕ or $-\phi$ with a fixed direction. If the isophote factors into two components corresponding to the choice of the sign, the isophote will be called decomposable. We will relate the decomposability of isophotes to the behaviour of the offset of the surface. The non-decomposable isophote is typically non-rational even on rational ruled surfaces, thus a genus formula will be derived. Section 4 will be devoted to the reconstruction of a ruled surface from a given collection of viewpoints together with the corresponding silhouettes in the planes of projection. This is a typical problem from an emerging area of algebraic vision, see e.g. Boyer and Berger (1997); Kang et al. (2001) for more papers on reconstruction of 3D shape from 2D information. It can be easily shown that a regular quadric is uniquely determined by three silhouettes. In the paper we will use the rationality of contour curves and we will prove a fact that two generic silhouettes are enough to reconstruct a rational ruled surface of degree at least three.

2. Preliminaries

If not stated otherwise, by a surface we mean an irreducible algebraic surface in the three-dimensional projective space. If $\mathcal{X}$ is such a surface, then $\mathcal{X}_{sm}$ denotes the set of its smooth points. For any fixed point $\mathbf{a} \in \mathbb{P}^3_{\mathbb{R}}$ the *contour* $\mathcal{C}_{\mathbf{a}}$ of $\mathcal{X}$ with respect to a viewpoint **a** is defined as the closure of the set

$$\{\mathbf{p} \in \mathcal{X}_{sm} : \mathbf{a} \in T_{\mathbf{p}}\mathcal{X}\}, \tag{1}$$

where $T_{\mathbf{p}}\mathcal{X}$ denotes the projective tangent plane at **p**. If $\mathcal{H}$ is an arbitrary plane not passing through **a** then we may project the contour $\mathcal{C}_{\mathbf{a}}$ from the point **a** to the plane $\mathcal{H}$. The projected curve is then the so called *silhouette* and we will denote it $\mathcal{S}_{\mathbf{a}}$, see Fig. 1.

Let $(x_0 : x_1 : x_2 : x_3)$ be coordinates in $\mathbb{P}^3_{\mathbb{R}}$. Fix a hyperplane $\omega : x_0 = 0$ and the absolute conic section $\Omega : x_0 = x_1^2 + x_2^2 + x_3^2$ then the complement $\mathbb{A}^3_{\mathbb{R}} = \mathbb{P}^3_{\mathbb{R}} \backslash \omega$ is an affine space endowed with the usual scalar product. The plane ω is called plane at infinity and its points can be understood as directions in $\mathbb{A}^3_{\mathbb{R}}$. We write $A = (a_1, a_2, a_3)$ for dehomogenization of a point $(1 : a_1 : a_2 : a_3)$ and $\overrightarrow{a} = (a_1, a_2, a_3)$ for dehomogenization of a direction $\mathbf{a} = (0 : a_1 : a_2 : a_3)$. Depending on the position of the point **a**, it is sometimes distinguished between the contour w.r.t. a central projection ($\mathbf{a} \notin \omega$) and a parallel projection ($\mathbf{a} \in \omega$).

The *Gauss mapping* $\gamma : \mathcal{X} \dashrightarrow (\mathbb{P}^3_{\mathbb{R}})^\vee$,[1] associated to a surface $\mathcal{X} \subset \mathbb{P}^3_{\mathbb{R}}$, assigns to a point of the surface its tangent plane $\gamma : \mathbf{p} \mapsto T_{\mathbf{p}}\mathcal{X}$, viewed as a point in the dual space $(\mathbb{P}^3_{\mathbb{R}})^\vee$. If $\mathcal{X}$ is given implicitly by a homogeneous polynomial equation $F(x_0, x_1, x_2, x_3) = 0$ then the formula for the Gauss mapping is just

$$\gamma : \mathbf{p} \mapsto \left(\partial_{x_0} F(\mathbf{p}) : \cdots : \partial_{x_3} F(\mathbf{p})\right). \tag{2}$$

[1] The dashed arrow emphasizes the fact that the mapping γ needs not to be defined for every point of the surface, but only on a dense subset.