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# Laplacian spectral basis functions

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ABSTRACT

Representing a signal as a linear combination of a set of basis functions is central in a wide range of applications, such as approximation, de-noising, compression, shape correspondence and comparison. In this context, our paper addresses the main aspects of signal approximation, such as the definition, computation, and comparison of basis functions on arbitrary 3D shapes. Focusing on the class of basis functions induced by the Laplace–Beltrami operator and its spectrum, we introduce the diffusion and Laplacian spectral basis functions, which are then compared with the harmonic and Laplacian eigenfunctions. As main properties of these basis functions, which are commonly used for numerical geometry processing and shape analysis, we discuss the partition of the unity and non-negativity; the intrinsic definition and invariance with respect to shape transformations (e.g., translation, rotation, uniform scaling); the locality, smoothness, and orthogonality; the numerical stability with respect to the domain discretisation; the computational cost and storage overhead. Finally, we consider geometric metrics, such as the area, conformal, and kernel-based norms, for the comparison and characterisation of the main properties of the Laplacian basis functions.



## 1. Introduction

In Computer Graphics, scalar functions are ubiquitous to represent the values of a physical phenomenon (e.g., the heat that diffuses from one or more source points), a shape descriptor (e.g., curvature, spherical harmonics), a distance (e.g., geodesic, bi-harmonic, diffusion distance) from a set of source points, or the pixels of a surface texture.

In all these cases, representing the input function in terms of a basis allows us to address a large number of applications, such as *multi-resolution representations* by selecting a set of multi-scale basis functions (e.g., Laplacian eigenfunctions, diffusion basis functions), *sparse representations* by choosing a low number of basis functions in order to achieve a target approximation accuracy, or *compression* by quantising the representation coefficients. We can also address *deformation* by modifying the coefficients that express the geometry of the input surface in terms of geometry-driven or shape-intrinsic basis functions, *smoothing* by neglecting the coefficients associated with large Laplacian frequencies, and the definition of *Laplacian spectral kernels and distances* as a filtered combination of the Laplacian spectral eigenfunctions.

In this context, our paper focuses on the main aspects of signal approximation on arbitrary 3D shapes, such as the definition, computation, and comparison of basis functions for applications in numerical geometry processing and shape

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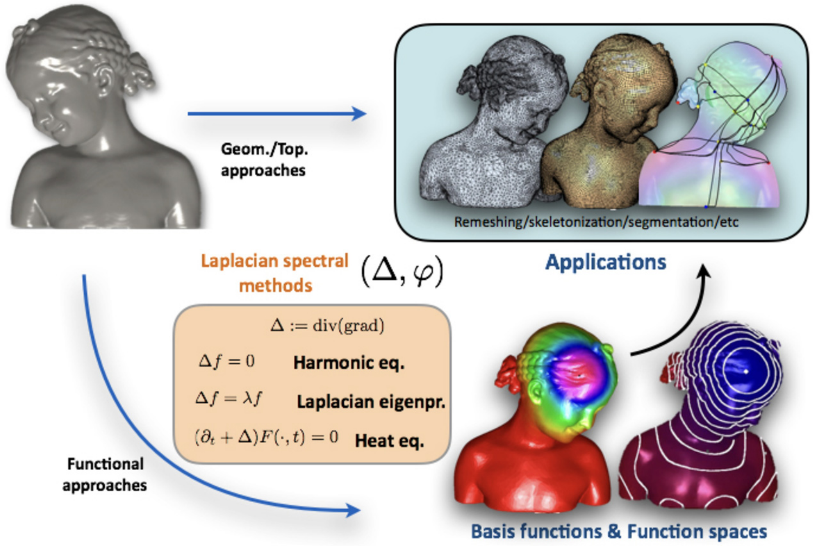


**Fig. 1.** Overview of the proposed approach for the definition of Laplacian spectral basis functions based on the solution to the harmonic equation, the Laplacian eigenproblem, and the diffusion equation, and on the filtering of the Laplacian spectrum.

analysis (Fig. 1). To define the functional space associated with an input 3D shape, we select a *set of its basis functions* and then represent any signal as a linear combination of these functions. In this work, we focus on the class of bases induced by the Laplace–Beltrami operator and its spectrum; i.e., the *harmonic basis functions* (Sect. 3), as solution to the Laplace equation; the *Laplacian eigenfunctions* (Sect. 4) associated with the spectrum of the Laplace–Beltrami operator; and the *Laplacian spectral basis functions* (Sect. 5), which are defined by filtering the Laplacian spectrum and include the *diffusion basis functions*.

The diffusion and Laplacian spectral functions are novel classes of functions, which will be defined by further developing our recent results on the Laplacian spectral distances (Patanè, 2017, 2016, 2014). Then, these functions will be compared with the harmonic functions and the Laplacian eigenfunctions. As a main contribution with respect to the previous work, the diffusion and Laplacian spectral basis functions are intrinsic to the input shape and local; i.e., they have a compact support that can be tuned easily by selecting the diffusion scale and the decay of the filter function to zero, respectively.

Defining basis functions with a compact support is particularly important to localise their behaviour around feature points and to reduce their storage overhead in the discrete setting. In fact, given a discrete domain $\mathcal{M}$ with $n$ points, the space of functions on $\mathcal{M}$ has dimension $n$ and its basis functions are encoded as a $n \times n$ matrix. This matrix can be stored only if we work with compactly-supported or analytically defined (e.g., radial basis or polynomial) functions, or if we select a subset of $k$ basis functions, with $k$ much smaller than $n$.

For the *comparison of functions* on the same surface (Sect. 6), we review different metrics (e.g., area, conformal, kernel-based metrics), which measure differential properties of the input scalar function and geometric properties of the underlying domain. As main *properties*, we discuss the partition of the unity and non-negativity; the intrinsic definition and invariance with respect to shape transformations (e.g., translation, rotation, uniform scaling); the locality, smoothness, and orthogonality; the numerical stability with respect to the domain discretisation; the computational cost and storage overhead.

Our experiments (Sect. 7, 8) show that the diffusion and the Laplacian spectral functions provide a valid alternative to the harmonic functions and Laplacian eigenfunctions. In fact, the diffusion basis functions can be centred at any point of the input domain, have a compact support, and encode local/global shape details according to the values of the temporal parameter or to the decay of the selected filter to zero. Similar properties also apply to the Laplacian spectral basis functions induced by a filter with a strong (e.g., exponential) decay to zero.

## 2. Laplace–Beltrami operator

Let $\mathcal{N}$ be a smooth surface, possibly with boundary, equipped with a Riemannian metrics and let us consider the *inner product* $\langle f, g\rangle_2 := \int_{\mathcal{N}} f(\mathbf{p})g(\mathbf{p})\mathrm{d}\mathbf{p}$ defined on the space $\mathcal{L}^2(\mathcal{N})$ of square integrable functions on $\mathcal{N}$ and the corresponding norm $\|\cdot\|_2$. The *Laplace–Beltrami operator* $\Delta := -\mathrm{div}(\mathrm{grad})$ is *self-adjoint* $\langle \Delta f, g\rangle_2 = \langle f, \Delta g\rangle_2$, and *positive semi-definite*; i.e., $\langle \Delta f, f\rangle_2 \geq 0$, $\forall f, g$ (Rosenberg, 1997). Finally, the value $\Delta f(\mathbf{p})$ does not depend on $f(\mathbf{q})$, for any couple of distinct points $\mathbf{p}$, $\mathbf{q}$ (*locality*).

According to Patanè (2016), we represent the Laplace–Beltrami operator on surface and volume meshes in a unified way as $\tilde{\mathbf{L}} := \mathbf{B}^{-1}\mathbf{L}$, where the *mass matrix* $\mathbf{B}$ is sparse, symmetric, positive definite, and the *stiffness matrix* $\mathbf{L}$ is sparse, symmetric, positive semi-definite, and $\mathbf{L1} = \mathbf{0}$. Analogously to the continuous case, the Laplacian matrix satisfies the following properties: