Computer Aided Geometric Design 64 (2018) 50–72

Contents lists available at ScienceDirect

Computer Aided Geometric Design

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# Rational fractal surface interpolating scheme with variable parameters ☆

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ARTICLE INFO

*Article history:*
Received 1 March 2018
Received in revised form 19 April 2018
Accepted 11 June 2018
Available online 25 June 2018

*Keywords:*
Iterated function systems
Bivariate rational FIFs
Function scaling factors
Fractal properties
Surface modeling

ABSTRACT

This paper presents a method of construction for Rational Fractal Surfaces (RFSs) on grids by using bivariate rational Fractal Interpolation Functions (FIFs) with function scaling factors. Further, analysis properties of bivariate rational FIFs are investigated, including the convergence, the stability and the box-counting dimension. Compared with the current interpolation, the developed bivariate rational interpolation can recapture the traditional non-recursive shape modifiable interpolation, and provide shape properties of the interpolant and fractality of the derivatives. Numerical examples show that the presented interpolation scheme can describe real phenomenon well.



## 1. Introduction

The concept of Fractal Interpolation Functions (FIFs) was first introduced by Barnsley (1986, 1988) using a certain Iterated Function System (IFS), and after that was extended to enable interpolation of fractal surfaces and three dimensional objects (Wittenbrink, 1995; Zhao, 1995). The fractal interpolation provides a powerful tool for dealing with the irregular data from real phenomena or data from a function with irregular derivatives, due to which gives good deterministic representations of complex phenomena. At present, fractal interpolating techniques have been widely used in many fields such as natural scenery simulation, graphics, image processing, securities market index analysis, physics and chemistry, structural mechanics and so on.

The use of IFS to represent surfaces has been explored by a number of authors and references. Massopust (1989, 1994) constructed the fractal interpolation surface (FIS) on triangular regions, where the interpolation points placed on the boundary of the domain is coplanar. However, this construction is not flexibility in modeling complex surfaces. Geronimo and Hardin (1993) generalized this construction to polygonal regions with arbitrary interpolation points and gave out the formula of the dimension. Zhao (1995) presented an approach to construct fractal surfaces by triangulation and proved theoretically that the attractors of this construction are continuous FISs. Xie and Sun (1997) investigated principles and mathematical models of bivariate fractal interpolation functions, and proved the theorem of the uniqueness of an iterated function system and the theorem of fractal dimensions of FISs. In addition, Dalla (2002) presented a method to construct

☆ This paper has been recommended for acceptance by Ligang Liu.
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https://doi.org/10.1016/j.cagd.2018.06.003
0167-8396/

FIFs on random grids by using collinear boundary points and demonstrated that the attractor is a continuous FIS. Malysz (2006) generalized Dalla's approach by using arbitrary boundary data, but the contraction factors for each map of the IFS were equal, and he generalized this construction to higher dimensions. Chand and Vijender (2015, 2016) discussed the visualization of positive surface and monotonity preserving rational quadratic fractal interpolation surfaces. Moreover, some properties such as smoothness, stability and convergence of the FIFs have been discussed in Bouboulis and Dalla (2007a) and Wang and Li (2008). And also, there have been many literatures generalizing fractal surface in different aspects. For example, Bouboulis et al. (2006) discussed recurrent bivariate fractal interpolation surfaces which generalized an affine FISs and computed their box-dimension. Bouboulis and Dalla (2007b) generalized the notion of FIFs to allow arbitrary data points lie on $\mathbb{R}^n$, and presented some relations between FIFs and the Laplace partial differential equation with Dirichlet boundary conditions. Qian (2002) constructed non-tensor product bivariate FIFs defined on gridded rectangular domains, and discussed the relevant Lagrange interpolation problem. Ruan and Xu (2015) introduced a special class of FISs which are called bilinear FISs defined on rectangular grids without any restriction on interpolation points and vertical scaling factors.

Due to some sort of self-similarity nature of FIFs, as a new type of interpolants, they have more advantages than the classical interpolants in fitting and approximation of naturally occurring functions. As important free parameters in the IFS, vertical scaling factors act a decisive influence on the properties of the FIF and shape of the fractal curve or surface. Almost all research work mentioned above are limited within the cases of constant scaling factors. The IFS with constant scaling factors is an effective tool for the description of the thing which has explicit self-similarity character. However, a perfect fractal does not exist in nature. Thus, it is difficult to accurately describe the real phenomenon by FIFs with constant scaling factors, and more serious, it might lead to obvious errors for approximating some complicated irregular data with less self-similarity. Hence, it is necessary to study FIFs generated by IFSs with variable scaling factors. And yet, to the best of our knowledge, it is not well explored hitherto, there is only a little work to contribute to the problem. Feng et al. (2012) proposed a method of construction for fractal interpolation surfaces on a rectangular domain with arbitrary interpolation nodes by using function vertical scaling factors. Ji and Peng (2013) presented a class of FIFs with vertical scaling factor functions and proved their some analytical properties. Yun et al. (2014) provided a method to construct fractal surfaces by recurrent fractal curves which were constructed by using a recurrent iterated functions system (RIFS) with function scaling factors. Further, Yun et al. (2015) constructed general RIFSs with function vertical scaling factors and prove the existence of bivariate functions whose graph are attractors of above-constructed RIFSs.

Most of FIFs with function scaling factors above are actually the fractal polynomial interpolation. It is well known, rational functions are more effective for describing complex phenomena than polynomial functions. In this paper, we construct a class of rational fractal interpolation surfaces with function vertical scaling factors, investigate some properties of bivariate rational fractal interpolation functions (BRFIFs), and estimate the lower and upper bound on the box-dimension of rational fractal surfaces. Due to variable scaling factors and embedded shape parameters in the BRFIFs, the shape of the rational fractal interpolation surfaces can be adjusted. The developed bivariate rational interpolation can recapture the traditional non-recursive shape modifiable interpolation, provide shape properties of the interpolant and fractality of the derivatives, and which is more flexible than the current interpolation.

A brief outline of this paper is as the following. In section 2, we first introduce the precise definition of a bivariate FIF generated by a certain IFS, and then propose a constructive method of rational fractal surfaces with function vertical factors based on developed bivariate rational Hermite interpolants with shape parameters. Section 3 discuss some properties of the BRFIFs. Section 4 estimates the upper and lower bounds on the box-counting dimension of rational fractal surfaces. In section 5, we present numerical examples to discuss and demonstrate the performance of the presented BRFIFs. Finally, some examples of practical applications are provided to evaluate the effectiveness of the constructed rational FIFs.

## 2. Rational fractal interpolation surfaces

### 2.1. Bivariate IFS on the rectangular grids

Let $\{(x_i, y_j, z_{i,j}),\ i = 1, 2, \cdots, N;\ j = 1, 2, \cdots, M\}$ be a given set of data points, and let $I = [a, b] \subseteq \mathbb{R}$ contains $\{x_0, x_1, \cdots, x_N\}$, $J = [c, d] \subseteq \mathbb{R}$ contains $\{y_1, y_2, \cdots, y_M\}$. Set $I_i = [x_i, x_{i+1}]$ for $i \in \mathscr{I} = \{1, 2, \cdots, N-1\}$, $J_j = [y_j, y_{j+1}]$ for $j \in \mathscr{J} = \{1, 2, \cdots, M-1\}$. Denote $\mathcal{I} = \{1, 2, \cdots, N-1\}$, $\mathcal{J} = \{1, 2, \cdots, M-1\}$. Let $L_i(x)$ be contractive homeomorphisms: $I \to I_i$:

$$L_i(x_1) = x_i,\quad L_i(x_N) = x_{i+1},$$

$$|L_i(c_1) - L_i(c_2)| \le \lambda |c_1 - c_2|,\quad \forall c_1, c_2 \in I,$$

where $0 \le \lambda < 1$. Let $\phi_j(y)$ be contractive homeomorphisms: $J \to J_j$:

$$\phi_j(y_1) = y_j,\quad \phi_j(y_M) = y_{j+1},$$

$$|\phi_j(d_1) - \phi_j(d_2)| \le \mu |d_1 - d_2|,\quad \forall d_1, d_2 \in J,$$

where $0 \le \mu < 1$. Further, let $F = I \times J \times \mathbb{R}$, and the continuous mappings $F_{i,j} : F \to \mathbb{R}$ fulfill: