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High order approximation to non-smooth multivariate functions ☆

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A B S T R A C T

Common approximation tools return low-order approximations in the vicinities of singularities. Most prior works solve this problem for univariate functions. In this work we introduce a method for approximating non-smooth multivariate functions of the form $f = g + r_+$ where $g, r \in C^{M+1}(\mathbb{R}^n)$ and the function r_+ is defined by

$$r_+(y) = \begin{cases} r(y), & r(y) \geq 0 \\ 0, & r(y) < 0 \end{cases} , \ \forall y \in \mathbb{R}^n .$$

Given scattered (or uniform) data points $X \subset \mathbb{R}^n$, we investigate approximation by quasi-interpolation. We design a correction term, such that the corrected approximation achieves full approximation order on the entire domain. We also show that the correction term is the solution to a Moving Least Squares (MLS) problem, and as such can both be easily computed and is smooth. Last, we prove that the suggested method includes a high-order approximation to the locations of the singularities.

1. Introduction

Approximation of non-smooth functions is a complicated problem. Common approximation tools, such as splines or approximations based on Fourier transform, return smooth approximations, thus relying on the smoothness of the original function for the approximation to be correct. However, the need to approximate non-smooth functions exists in many applications. For a high-order approximation of non-smooth functions, we need to allow our approximation to be non-smooth. Otherwise, in the vicinities of the singularities, we will get a low-order approximation. In this work we will suggest a method that will allow us to properly approximate non-smooth functions of a given model.

We will concentrate on functions $f : \mathbb{R}^n \to \mathbb{R}$ which may be modelled as $f = g + r_+$ where $g, r \in C^{M+1}(\mathbb{R}^n)$ and the function r_+ is defined by

$$r_+(y) = \begin{cases} r(y), & r(y) \geq 0 \\ 0, & r(y) < 0 \end{cases} , \ \forall y \in \mathbb{R}^n .$$

Such functions are obviously continuous, but are non-smooth across the hypersurface

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$$\Gamma_r := \left\{ z \in \mathbb{R}^n : r(z) = 0 \right\} .$$

As an example for such functions, consider shock waves, which are solutions of non-linear hyperbolic PDEs (Morse and Ingard, 1986). Another example would be a signed distance function (Osher and Fedkiw, 2003), where the distance is measured from a disconnected set. Our goal is to achieve high-order approximations of such functions. To achieve that we will concentrate on a specific family of approximation tools.

Consider a quasi-interpolation operator Q (Wendland, 2004). Such an operator receives the values of a function $\phi : \mathbb{R}^n \to \mathbb{R}$ on a set of data points $X \subset \mathbb{R}^n$. The quasi-interpolation operator Q returns an approximation defined by

$$Q\phi(y) := \sum_{x \in X} q_x(y)\phi(x) \ , \ \forall y \in \mathbb{R}^n \ ,$$

where $\{q_x\}$ are the quasi-interpolation basis functions, each is smooth and has compact support.

Let h be the fill distance of X,

$$h := \min \left\{ L : B_L(y) \cap X \neq \emptyset \ , \ \forall y \in \mathbb{R}^n \right\} \ ,$$

where $B_r(y)$ is the ball of radius r centred at y. Denote

$$\Upsilon_M := \min \left\{ L > 0 : \forall y \in \mathbb{R}^n \ , \ B_{Lh}(y) \cap X \text{ is uni-solvent for } \Pi_M(\mathbb{R}^n) \right\} .$$

Here,

$$\Pi_M(\mathbb{R}^n) := \left\{ p : \mathbb{R}^n \to \mathbb{R} : deg(p) \le M \right\},$$

and $deg(p)$ is the total degree of the polynomial p. Thus, h is the minimal radius which is guaranteed to contain a data point, and $\Upsilon_M \cdot h$ is the minimal radius that guarantees enough data points to uniquely determine each polynomial in $\Pi_M(\mathbb{R}^n)$. We will also assume that there exists $N > 0$ such that for all $y \in \mathbb{R}^n$ we have

$$\frac{\#\left(X \cap B_h(y)\right)}{h^n} \le N \ .$$

That is, the data set X has no accumulation points. Denote

$$R := min \left\{ \rho > 0 : supp(q_x) \subseteq B_{\rho h}(x) \ , \ \forall x \in X \right\} \ . \tag{1}$$

We assume that the operator Q has a bounded Lebesgue constant

$$L_1 := \sup \left\{ \sum_{x \in X} |q_x(y)| : y \in \mathbb{R}^n \right\} \tag{2}$$

and reproduces polynomials in $\Pi_M(\mathbb{R}^n)$. Then, the error in the quasi-interpolation,

$$E\phi := \phi - Q\phi$$

satisfies for all $\phi \in C^{M+1}(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$

$$|E\phi(y)| \le C_1 \cdot \|\phi\|_{C^{M+1}} \cdot h^{M+1}$$

where

$$C_1 = (1 + L_1) \cdot R^{M+1}$$

and

$$\|\phi\|_{C^{M+1}} := \sum_{|\beta| = M+1} \frac{\|D^\beta \phi\|_\infty}{\beta!}$$

with β a multi-index and $\| \cdot \|_\infty$ the maximum norm. That is, the operator Q has full approximation order for smooth functions (Wendland, 2004). On the other hand, since the approximation $Q\phi$ is always smooth, the operator gives low-order approximations in the vicinities of singularities.

One example of a quasi-interpolation operator is the MLS approximation (Bos and Salkauskas, 1989; Levin, 1998). Given a function $\phi : \mathbb{R}^n \to \mathbb{R}$ and a point $y \in \mathbb{R}^n$ the MLS approximation is defined as $Q\phi(y) := p_y(y)$ where

$$p_y := \underset{p \in \Pi_M(\mathbb{R}^n)}{\arg\min} \sum_{x \in X} \eta\left(\frac{\|y - x\|}{h}\right) \cdot (p(x) - \phi(x))^2 \ . \tag{3}$$