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Support function at inflection points of planar curves

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ABSTRACT

We study the support function in the neighborhood of inflections of oriented planar curves. Even for a regular curve, the support function is not regular at the inflection and is multivalued on its neighborhood. We describe this function using an implicit algebraic equation and the rational Puiseux series of its branches. Based on these results we are able to approximate the curve at its inflection to any desired degree by curves with a simple support function, which consequently possess rational offsets.
We also study the G^1 Hermite interpolation at two points of a planar curve. It is reduced to the functional C^1 interpolation of the support function. For the sake of comparison and better understanding, we show (using standard methods) that its approximation order is 4 for inflection-free curves. In the presence of inflection points this approximation is known to be less efficient. We analyze this phenomenon in detail and prove that by applying a nonuniform subdivision scheme it is possible to receive the best possible approximation order 4, even in the inflection case.

1. Introduction

The support function representation describes a curve as the envelope of its tangent lines, where the distance between the line and the origin is specified by a function of the line unit normal vector. This approach is one of the classical tools in convex geometry (Gruber and Wills, 1993). In this representation, offsetting and convolution of curves correspond to simple algebraic operations of the corresponding support functions. In addition, it provides a computationally simple way to extract curvature information (Gravesen, 2007). Applications of this representation to problems from Computer Aided Design were foreseen in the classical paper (Sabin, 1974) and developed in several recent publications, see e.g., Aigner et al. (2009a, 2009b), Bastl et al. (2012), Gravesen et al. (2007, 1993), Lávička et al. (2010), Šír et al. (2010, 2008).

One of the main advantages of this representation consists in the possibility of handling efficiently both the shape and its offsets (or more generally its convolutions with some other shapes). Indeed, in this representation the approximation error remains the same for the curve/surface and its offsets; in addition the rational parameterization can be ensured for both simultaneously (Šír et al., 2008). This fact can be applied in the context of CNC manufacturing, where the tool center does not follow the produced object boundary, but rather a convolution with the tool shape. The use of a support function representation allows one to avoid complicated approximation techniques in the offset or convolution description.

Another advantage of using the support function is a rather simple solution of the G^1 Hermite interpolation. Geometric interpolations (where not exact tangent vectors but only their directions are interpolated) typically lead to nonlinear

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equations, see e.g., Meek and Walton (1997). Via the support function representation, however, it is reduced to the usual functional interpolation and a system of linear equations is obtained (Blažková and Šír 2014a, 2014b; Šír et al., 2010).

Probably the main difficulty related to this kind of representation is the fact that the support function typically has single values for convex objects or, more generally, for so-called quasi-convex shapes (Šír et al., 2008). In general, it is multivalued. A certain effort was made in an attempt to handle the multivalued nature of the support function, considering its implicit version (Lávička et al., 2010) or trigonometric polynomials with various rational periods (Šír et al., 2010).

For the purpose of this paper an inflection is a point where the tangent meets the curve to an order of at least 3. So far, however, inflections have been excluded from the description using the support function. The main reason is that the normal turns back at inflections, and the support function must be considered as multivalued in its neighborhood. It has also been noted (Bastl et al., 2013; Šír et al., 2010) that in the presence of inflections, the G^1 Hermite interpolation is less efficient.

The purpose of this paper is to analyze the behavior of the support function in the neighborhood of a curve inflection and use it to design an applicable interpolation scheme. We obtain a description of the support function at inflections using the rational Puiseux series of its branches. Based on these results we are able to approximate the curve at its inflection to any desired degree by curves with a simple support function, which consequently possess rational offsets. We also study the G^1 Hermite interpolation at two points of a planar curve. We prove that its approximation order is 4 for inflection-free curves. In the presence of inflection points this approximation is known to be less efficient. We analyze this phenomenon and, applying a nonuniform subdivision scheme, we receive the best possible approximation order 4, even in the inflection case.

The remainder of the paper is organized as follows. In Section 2 we give the basic definitions and properties of the explicit and implicit support function. We also analyze the behavior of the support function at inflections. In Section 3 we revisit the topic of the G^1 Hermite interpolation and show that the approximation order drops from 4 to 3 when an inflection occurs. Then we show that the nonuniform subdivision is sufficient to restore the approximation order 4. In Section 4 we design an approximation algorithm and illustrate the theoretical results on some examples. Finally, we conclude the paper.

2. Support function representation of curves

In this section we first recall the basic definitions and properties of the dual support function representation of planar curves. In the second part we analyze the behavior of this function at the inflection points of the primary curve.

2.1. Basic definitions and properties

For a parametrically or implicitly given oriented planar curve $\mathcal{C}$ we define the support function h as a (possibly multivalued) function defined on (a subset of) the unit circle

$$h : \mathbb{S}^1 \supset U \to \mathbb{R}$$

which to a unit normal $\mathbf{n} = (n_1, n_2)$ associates the oriented distance(s) from the origin $[0, 0]$ to the corresponding tangent line(s) of the curve.

As shown e.g. in Šír et al. (2008) we can recover the curve $\mathcal{C}$ from $h(\mathbf{n})$ as the envelope of the system of the tangent lines $\{\mathbf{n} \cdot \mathbf{x} - h(\mathbf{n}) = 0 : \mathbf{n} \in U\}$. This envelope is locally parameterized via the formula

$$\mathbf{c}_h(\mathbf{n}) = h(\mathbf{n})\mathbf{n} + \nabla_{\mathbb{S}_1} h(\phi) = h(\mathbf{n})\mathbf{n}(\phi) + h'(\phi)\mathbf{n}'(\phi), \tag{1}$$

where $\nabla_{\mathbb{S}_1}$ denotes the intrinsic gradient with respect to the unit circle, which is alternatively expressed using the following arc length parameterization of $\mathbb{S}_1$

$$\mathbf{n}(\phi) = (\cos(\phi), \sin(\phi)), \qquad \mathbf{n}'(\phi) = \mathbf{n}^{\perp}(\phi) = (-\sin(\phi), \cos(\phi)). \tag{2}$$

It is also possible to parameterize birationally the unit circle, for example via

$$\mathbf{n} = [n_1, n_2] = \left[\frac{1 - s^2}{1 + s^2}, \frac{2s}{1 + s^2}\right] \tag{3}$$

and obtain a kind of affine version of the support function $h(s)$ together with the parameterization of the curve $\mathbf{c}_h(s)$. Note that the inversion formula for changing the variables is

$$s = \frac{n_2}{1 + n_1} = \arctan(\phi/2).$$

As there are often many tangent lines with the same normal, it is globally not always possible to obtain an explicit expression of h but rather an implicit one, which is closely related to the notion of a dual curve consisting of the tangent lines of $\mathcal{C}$.