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Short Communication

Log-aesthetic curves as similarity geometric analogue of Euler's elasticae ☆

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ABSTRACT

In this paper we consider the log-aesthetic curves and their generalization which are used in CAGD. We consider those curves under similarity geometry and characterize them as stationary integrable flow on plane curves which is governed by the Burgers equation. We propose a variational formulation of those curves whose Euler–Lagrange equation yields the stationary Burgers equation. Our result suggests that the log-aesthetic curves and their generalization can be regarded as the similarity geometric analogue of Euler's elasticae.

1. Introduction

In this paper we consider a class of plane curves in CAGD called the *log-aesthetic curves* (LAC) and their generalization called the *quasi aesthetic curves* (qAC), and present their new mathematical characterization by using the theory of integrable systems. In 1744, Euler introduced the elastic energy of a plane curve which is the totality of squared signed curvature, and studied a variational problem of plane curves in Euclidean geometry (Euler, 1744). The solutions to this problem are called *elasticae*. On the other hand, a standard isoperimetric deformation of arc length parametrized plane curves is governed by the *modified Korteweg de Vries (mKdV) equation* for the signed curvature (Goldstein and Petrich, 1991), which is one of the most important integrable equations. Then it was seen that elasticae are characterized as travelling wave solutions to the mKdV equation. In this paper, we consider LAC and qAC in the framework of *similarity geometry*, and characterize them as the similarity geometric analogue of elasticae. We formulate them as stationary curves with respect to the simplest integrable flow given by the *Burgers equation*. We then give a variational formulation by introducing a functional (called the fairing energy) for plane curves, from which we deduce the stationary Burgers equation as the Euler–Lagrange equation.

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This gives a new mathematical characterization of LAC and qAC and demonstrates the usefulness of Klein geometry and integrable systems as tools of CAGD.

2. Elasticae and mKdV flows

To clarify the mathematical background and motivations of this paper, we recall briefly here the basic facts on the elasticae and the Goldstein–Petrich flows. An arc length parametrized curve $\gamma(s)$ in the Euclidean plane $\mathbb{E}^2$ is said to be an *elastica* if it is a critical point of the *elastic energy*:

$$E(\gamma) = \int_0^\ell \frac{1}{2}\kappa(s)^2\, ds, \tag{1}$$

through total length preserving variations. Here s is the arc length parameter and $\kappa(s)$ is Euclidean (signed) curvature. We consider the smooth plane curves $\gamma : [0,\ell] \to \mathbb{E}^2$ with two endpoints and tangent vectors at the ends being fixed. Then the Euler–Lagrange equation obtained from the variation of the curve is

$$2\kappa'' + \kappa^3 - \lambda\kappa = 0, \tag{2}$$

where λ is a constant and $' = d/ds$. For more information on the elasticae, we refer to (Bryant and Griffiths, 1986; Mumford, 1994).

On the other hand, we consider the isoperimetric deformation of plane curves of the form (*Goldstein–Petrich flow* Goldstein and Petrich, 1991, see also Lamb, 1976):

$$\dot{\gamma} = -\kappa' N^{\mathsf{E}} - \frac{1}{2}\kappa^2 T^{\mathsf{E}}.$$

Here $\dot{} = \partial/\partial t$, $T^{\mathsf{E}} = \gamma'$ and $N^{\mathsf{E}} = JT^{\mathsf{E}}$, where J is the positive $\pi/2$-rotation, T^{E} and N^{E} are the unit tangent vector field and the unit normal vector field, respectively. The Frenet frame $F^{\mathsf{E}} = (T^{\mathsf{E}}, N^{\mathsf{E}}) \in \mathrm{SO}(2)$ satisfies

$$(F^{\mathsf{E}})^{-1} F^{\mathsf{E}\prime} = \kappa J, \quad (F^{\mathsf{E}})^{-1} \dot{F}^{\mathsf{E}} = -\left(\kappa'' + \frac{1}{2}\kappa^3\right) J, \tag{3}$$

where the first equation is the Frenet formula. The compatibility condition of (3) yields the mKdV equation

$$\dot{\kappa} + \kappa''' + \frac{3}{2}\kappa^2\kappa' = 0. \tag{4}$$

In the terminology of integrable systems, (3) is nothing but a *Lax pair* (for vanishing spectral parameter). The travelling wave solutions $\kappa(s - \lambda t/2)$ to the mKdV equation satisfy $2\kappa'' + \kappa^3 - \lambda\kappa = c_0$ for some constant $c_0 \in \mathbb{R}$. Comparing with (2), we see that every elastica is a solution to the Goldstein–Petrich flow.

3. Log-aesthetic curves and similarity geometry

According to (Miura, 2006), an arc length parametrized plane curve $\gamma(s) \in \mathbb{R}^2$ is said to be an LAC of slope α if its signed curvature radius q satisfies

$$q(s)^\alpha = as + b \ \ (\alpha \neq 0), \quad q(s) = \exp(as + b) \ \ (\alpha = 0), \quad a, b \in \mathbb{R}. \tag{5}$$

The class of LAC's includes some well known plane curves, for instance, the logarithmic spiral ($\alpha = 1$), the clothoid ($\alpha = -1$) and the *Nielsen spiral* ($\alpha = 0$). Fig. 1 illustrates some examples of LAC.

Recent advancements on the LAC have been promising as indicated by Levien and Séquin (Levien, 2009). Theoretical backgrounds have been set up, such as variational formulation of LAC for free-form surface design (Miura et al., 2012), and fast computational algorithm (Ziatdinov et al., 2012). An important development for industrial design practices has been presented in (Meek et al., 2012), where it is proved that a unique solution exists for the G^1 interpolation problem using an LAC segment when $\alpha < 0$.

Those developments have been made based on the basic characterization (5) in the framework of Euclidean geometry. However, (5) is too simple to identify the underlying geometric structure. Consequently, we do not have a “good” guideline to generate a larger class of “aesthetic” geometric objects including LAC based on a sound mathematical background. In this paper, we clarify that LAC fits well the framework of the similarity geometry.

The similarity plane geometry is a Klein plane geometry associated with the group of similarity transformations, *i.e.*, isometries and scalings: $\mathbb{R}^2 \ni \boldsymbol{p} \longmapsto rA\boldsymbol{p} + \boldsymbol{b}$, $A \in \mathrm{SO}(2)$, $r \in \mathbb{R}_+$, $\boldsymbol{b} \in \mathbb{R}^2$. The natural parameter of plane curves in similarity geometry is the *turning angle* $\theta = \int \kappa(s)ds$. Let $\gamma(\theta)$ be a plane curve in similarity geometry parametrized by θ. The *similarity*