Computer Aided Geometric Design 60 (2018) 10–17

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

Symmetric four-directional bivariate pseudo-spline symbols ☆

Costanza Conti [a,*], Chongyang Deng [b], Kai Hormann [c]

[a] Dipartimento di Ingegneria Industriale, Università di Firenze, Viale Morgagni, 40/44 - 50134 Firenze, Italy
[b] School of Science, Hangzhou Dianzi University, Hangzhou, 310018, China
[c] Faculty of Informatics, Università della Svizzera italiana, 6900 Lugano, Switzerland

ARTICLE INFO

Article history:
Received 13 June 2017
Received in revised form 11 November 2017
Accepted 8 January 2018
Available online xxxx

Keywords:
Box spline
Symbol
Subdivision
Polynomial generation/reproduction
Approximation order
Pseudo-splines

ABSTRACT

Univariate pseudo-splines are a generalization of uniform B-splines and interpolatory $2n$-point subdivision schemes. Each pseudo-spline is characterized as the limit of the subdivision scheme with least possible support among all schemes with specific degrees of polynomial generation and reproduction. In this paper we propose a formula for the symbols of the bivariate counterpart of pseudo-splines.

1. Introduction

Univariate pseudo-splines (Dong and Shen, 2007) are the limits of subdivision schemes with least possible support among all schemes with specific degrees of polynomial generation and reproduction, and they neatly fill the gap between uniform B-splines and interpolatory $2n$-point schemes. As a first step towards the generalization of this concept to the bivariate setting, we propose a family of symmetric four-directional bivariate symbols $a_n^l(\boldsymbol{z})$, $0 \le l < n$ and prove that the members of this family satisfy the algebraic properties for polynomial generation and reproduction up to degree $2n-1$ and $2l+1$, respectively. We further show that the special cases $a_n^0(\boldsymbol{z})$ and $a_n^{n-1}(\boldsymbol{z})$ are the symbols of the four-directional box splines and the minimally supported interpolatory schemes by Han and Jia (1998), respectively. Hence, our family fills the gap between these schemes, akin to univariate pseudo-splines.

All methods we use are of purely algebraic nature and work directly on the bivariate subdivision *symbol* defined by the finitely supported *subdivision mask* $A = \{a_{\boldsymbol{\alpha}} \in \mathbb{R} : \boldsymbol{\alpha} \in \mathbb{Z}^2\}$ as

$$a(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^2} a_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}}, \qquad \boldsymbol{z}^{\boldsymbol{\alpha}} = z_1^{\alpha_1} z_2^{\alpha_2} \qquad \boldsymbol{z} = (z_1, z_2) \in (\mathbb{C} \setminus \{0\})^2.$$

In the four-directional setting that we consider, the symbol is called *symmetric* if

$$a(z_1, z_2) = a(1/z_1, z_2) = a(z_1, 1/z_2) = a(z_2, z_1).$$

☆ This paper has been recommended for acceptance by Ulrich Reif.
* Corresponding author.

E-mail addresses: costanza.conti@unifi.it (C. Conti), dcy@hdu.edu.cn (C. Deng), kai.hormann@usi.ch (K. Hormann).

https://doi.org/10.1016/j.cagd.2018.01.001

The *support* of the mask, the symbol, and the scheme is defined as the convex hull of the set $\{\boldsymbol{\alpha} \in \mathbb{Z}_2 : a_{\boldsymbol{\alpha}} \neq \mathbf{0}\}$, and the *size* of the support is the area of this convex hull.

Similar to the univariate setting, the generation and reproduction degrees of a bivariate subdivision scheme are closely related to the behavior of the symbol and its derivatives at $\boldsymbol{z} \in E$, where $E = \{(1,1), (-1,1), (1,-1), (-1,-1)\}$. For example, the generation and reproduction of constant functions is guaranteed, if $a(1,1) = 4$ and $a(\boldsymbol{z}) = 0$ for $\boldsymbol{z} \in E'$, where $E' = E \setminus \{(1,1)\}$, which in turn is a necessary condition for the convergence of the scheme. Regarding higher degrees of generation and reproduction, Cavaretta et al. (1991) show that a convergent bivariate scheme *generates* polynomials up to degree $m \geq 1$, if

$$(D^{\boldsymbol{k}} a)(\boldsymbol{z}) = 0, \qquad \boldsymbol{z} \in E', \quad \boldsymbol{k} \in \mathbb{N}_0^2, \quad 0 \leq |\boldsymbol{k}| \leq m, \tag{1}$$

which is also known as the *sum rule of order* $m+1$. Moreover, Charina and Conti (2013) prove that a non-singular primal scheme that generates polynomials up to degree m further *reproduces* polynomials up to degree $m' \geq 1$ with $m' \leq m$, if

$$(D^{\boldsymbol{k}} a)(1,1) = 0, \qquad \boldsymbol{k} \in \mathbb{N}_0^2, \quad 0 < |\boldsymbol{k}| \leq m'. \tag{2}$$

In Sauer (2002), Sauer shows that

$$\mathcal{J}_k = \left\langle 1 - \boldsymbol{z}^2 \right\rangle^k = \left\langle (1 - z_1^2)^{\alpha_1} (1 - z_2^2)^{\alpha_2} : \boldsymbol{\alpha} \in \mathbb{N}_0^2, |\boldsymbol{\alpha}| = k \right\rangle, \qquad k \geq 1, \tag{3}$$

is the ideal of all bivariate polynomials p which satisfy

$$(D^{\boldsymbol{k}} p)(\boldsymbol{z}) = 0, \qquad \boldsymbol{z} \in E, \quad \boldsymbol{k} \in \mathbb{N}_0^2, \quad 0 \leq |\boldsymbol{k}| < k,$$

and that the bivariate polynomials which satisfy only (1) for $m = k-1$ belong to the quotient ideal

$$\mathcal{I}_k = \mathcal{J}_k : \langle 1 - \boldsymbol{z} \rangle^k, \qquad k \geq 1. \tag{4}$$

Consequently, a convergent scheme with symbol $a \in \mathcal{I}_k$ generates polynomials up to degree $k-1$. However, $a \in \mathcal{J}_k$ does not imply polynomial reproduction of degree $k-1$, because $a(1,1) = 0$ in this case, and hence such a scheme is not even convergent (Dyn and Levin, 2002). But if a reproduces polynomials up to degree $k-1$ and $b \in \mathcal{J}_k$, then the reproduction degree of $a+b$ is also $k-1$. Note that the indices of our versions of $\mathcal{I}_k$ in (4) and $\mathcal{J}_k$ in (3) are shifted by one with respect to those in Sauer (2002) for convenience, so that

$$a \in \mathcal{I}_k, \quad b \in \mathcal{I}_l \qquad \Longrightarrow \qquad a \cdot b \in \mathcal{I}_{k+l},$$

and similarly for $\mathcal{J}_k$.

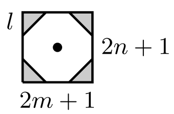
In what follows it will be useful to define the bivariate analogues of $\sigma = \frac{(1+z)^2}{4z}$ and $\delta = \frac{(1-z)^2}{4z}$, their difference, and their product as

$$\boldsymbol{\sigma}(\boldsymbol{z}) = \sigma(z_1)\sigma(z_2), \qquad \boldsymbol{\delta}(\boldsymbol{z}) = \delta(z_1)\delta(z_2), \qquad \boldsymbol{\gamma}(\boldsymbol{z}) = \boldsymbol{\sigma}(\boldsymbol{z}) - \boldsymbol{\delta}(\boldsymbol{z}), \qquad \boldsymbol{\pi}(\boldsymbol{z}) = \boldsymbol{\sigma}(\boldsymbol{z})\boldsymbol{\delta}(\boldsymbol{z}).$$

We further introduce the notation

$$\boldsymbol{\pi}(\boldsymbol{z})^{\boldsymbol{\alpha}} = \big(\sigma(z_1)\delta(z_1)\big)^{\alpha_1} \big(\sigma(z_2)\delta(z_2)\big)^{\alpha_2} = \frac{\delta(z_1^2)^{\alpha_1} \delta(z_2^2)^{\alpha_2}}{4^{\alpha_1+\alpha_2}} \in \mathcal{J}_{2|\boldsymbol{\alpha}|} \subset \mathcal{I}_{2|\boldsymbol{\alpha}|}. \tag{5}$$

Besides the degrees of polynomial generation and reproduction, we are also interested in the *support of a symbol*, and we frequently use the graphical notation



to denote the octagonal region $\{\boldsymbol{\alpha} : |\alpha_1| \leq m, |\alpha_2| \leq n, |\alpha_1| + |\alpha_2| \leq m+n-l\}$, or rather the rectangle $[-m,m] \times [-n,n]$, minus the triangular regions with side length l in each corner.

Following these conventions, we write the symbol of a *primal symmetric four-directional box spline as*

$$B_{j,k}(\boldsymbol{z}) = \left(\frac{1+z_1}{2}\right)^{2j} \left(\frac{1+z_2}{2}\right)^{2j} \left(\frac{1+z_1 z_2}{2}\right)^{k} \left(\frac{1+z_1/z_2}{2}\right)^{k} \frac{1}{z_1^{j+k} z_2^{j}} = \boldsymbol{\sigma}(\boldsymbol{z})^j \boldsymbol{\gamma}(\boldsymbol{z})^k \tag{6}$$

and recall from Charina et al. (2011) *that this symbol is contained in* $\mathcal{I}_{2m}$, *where* $m = 2j + k - \max(j,k)$. *Note that for given* m *the support*