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Numerically robust computation of circular visibility ☆

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ABSTRACT

We address the question of whether a point inside a domain bounded by a simple closed arc spline is circularly visible from a specified arc from the boundary. We provide a simple and numerically stable linear time algorithm that solves this problem. In particular, we present an easy-to-check criterion that implies that a point is not visible from a specified boundary arc.

1. Introduction

A point in the plane is called *circularly visible* from another point inside a planar domain if the two points can be connected by a circular arc that lies inside this domain. Algorithms that compute the set of all circularly visible points inside a polygon from a point or edge are well studied, cf. Agarwal and Sharir (1993), Chou et al. (1992), Chou and Woo (1995). In Agarwal and Sharir (1993), circular visibility from a point inside a simple polygon is treated. The authors present an $O(n \log n)$ algorithm, with n the number of vertices of the polygon, that computes the set of all circularly visible points. In Chou and Woo (1995) an algorithm to compute the circular visibility set of a point inside a simple polygon is presented, which is based on the so called CVD (circular visibility diagram), a partition of the plane where every point represents the center of an arc. This leads to an algorithm with linear runtime with respect to the number of vertices. A discussion of numerical stability is not existing in both cases, but numerical problems can be assumed if relevant circular arcs are almost straight. The computation of the CVD is further used to compute the circular visibility set from an edge of a simple polygon in Chou et al. (1992). The runtime of the presented algorithm is $O(kn)$ where n is the number of vertices and k is the number of CVDs computed which equals n in worst case. The problem we tackle here differs in two ways: we want to consider domains bounded by an *arc spline*, a curve that consists of circular arcs and line segments, and we only want to know if a point is visible from a specified arc on the boundary. The treatment of regions bounded by arc splines has not been considered yet in literature. We present a simple and numerically stable algorithm that decides, in linear time with respect to the number of arc segments, if a point is circularly visible from a boundary arc. For this purpose, we supply an easy-to-check criterion that directly implies that a point is not circularly visible from an arc. Although we only consider circular visibility of a point, we compute in some sense extremal arcs having a so-called alternating sequence. This enables

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that this approach can easily be extended to compute boundary arcs of the circular visibility set. This is a nice property as you are usually most interested in this boundary region.

We use this algorithm to improve the numerical stability of the SMAP (smooth minimum arc path) approach which computes an approximating smooth arc spline with the minimal number of segments within a specified maximal tolerance, cf. Maier (2014) or Schindler (2013) for an application in vehicle self-localization. The basic task in the SMAP algorithm is closely related to the computation of the circular visibility set from a starting arc. It is known that the boundary of the circular visibility set consists of "boundary arcs" having three points in common with the boundary of the domain. Due to even very small numerical inaccuracies, however, such boundary arcs can be missed. With the approach presented in this paper, we can determine if a point is visible and so we can localize the position of boundary arcs.

This paper is organized as follows: In Section 2, we introduce basic notations and definitions. In Section 3, we define a key tool for later proofs, a total order on a specified set of arcs. In Section 4, a sufficient condition for a point to be not circularly visible from an arc is shown. In Section 5, we present a linear time algorithm to decide if a point is circularly visible from an arc.

2. Notation and basic definitions

We call a continuous mapping $\alpha : [0,1] \to \mathbb{R}^2$ a *path* and $\alpha(0)$ its *starting point* and $\alpha(1)$ its *endpoint*. A path α is *closed* if $\alpha(0) = \alpha(1)$, it is *simple* if it is injective and *simple closed* if it is closed and $\alpha\big|_{[0,1)}$ is injective. Note that the image of a simple closed path is a *Jordan curve* which divides its complement, according to the Jordan curve theorem, into two connected components: a bounded one which we call the *interior* of the Jordan curve and an unbounded one, its *exterior*. As usual in the literature, we will use α for both the mapping and the image, usually referred to as a *curve*. In particular, this allows us to write $p \in \alpha$ instead of $p \in \alpha([0,1])$.

We denote $\alpha((0,1))$ by α° and by $\overline{\alpha}$ the *reverse path* defined by $\overline{\alpha}(t) = \alpha(1-t)$. Let α be a simple path and $p \in \alpha$. We denote by $t_\alpha(p)$ the unique parameter in $[0,1]$ with $\alpha(t_\alpha(p)) = p$. We write $t(p)$ if the corresponding path is clear from the context. For $p, q \in \alpha$ we write $p \prec_\alpha q$ if $t_\alpha(p) < t_\alpha(q)$.

A path γ of the form

$$\gamma(t) = c + r \cdot \begin{pmatrix} \cos(2\pi a t + t_1) \\ \sin(2\pi a t + t_1) \end{pmatrix}, \qquad c \in \mathbb{R}^2, r > 0, a \in (0,1), t_1 \in [0, 2\pi),$$

is called a *positively oriented arc*. We call the reverse path $\overline{\gamma}$ of a positively oriented arc a *negatively oriented arc*. The path ℓ defined by $\ell(t) = (1-t)\cdot p_1 + t \cdot p_2,\ p_1, p_2 \in \mathbb{R}^2, p_1 \neq p_2$, is a *line segment from p_1 to p_2* denoted by $[p_1, p_2]$. We call a path an *arc* if it is an arc of either orientation or a line segment. The set of all arcs will be denoted by Γ.

As an arc γ is differentiable with respect to t and its derivative $\dot{\gamma}(t)$ does not vanish for any $t \in [0,1]$, we can define the *unit tangent vector* $\gamma' : [0,1] \to S^1$, where S^1 is the unit sphere, by $\gamma'(t) := \frac{\dot{\gamma}(t)}{||\dot{\gamma}(t)||_2}$ and the *normal of length one "to the left"* $\gamma^\perp(t) := (-v, u)^T$ with $(u, v)^T = \gamma'(t)$.

For $p, q, r \in \mathbb{R}^2$, $\tau \in S^1$, we denote by $\gamma[p, r, q]$ the arc with starting point p, endpoint q that passes through r and by $\gamma[\tau, p, q]$ the arc with starting point p, endpoint q and τ as starting point tangent and by $\gamma[p, q, \tau]$ the arc with starting point p, endpoint q and τ as endpoint tangent. Note that $\gamma[p, r, q]$ exists and is unique if p, q, r are distinct, $q \notin [p, r]$ and $p \notin [r, q]$. Likewise, $\gamma[\tau, p, q]$ and $\gamma[p, q, \tau]$ exist and are unique if $p \neq q$ and τ and $(p-q)$ are not pointing into the same direction.

Let γ be a positively or negatively oriented arc or a line segment, then we call $[\gamma] := \gamma(\mathbb{R})$ the *corresponding circle* or the *corresponding line*, respectively. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be arcs with $\gamma_k(1) = \gamma_{k+1}(0)$, $k \in \{1, \ldots n-1\}$. Then, we call the path $\gamma_1 \sqcup \gamma_2 \sqcup \ldots \sqcup \gamma_n$ defined as the concatenation

$$(\gamma_1 \sqcup \gamma_2 \sqcup \cdots \sqcup \gamma_n)(t) := \gamma_k(nt - k + 1),\ t \in \frac{1}{n}[k-1, k],\ k = 1, 2, \ldots, n$$

an *arc spline with n segments*. We call an arc spline *simple*, *closed* or *simple closed* if the corresponding path is simple, closed or simple closed, respectively. The points $\gamma_1(0), \gamma_2(0), \ldots, \gamma_n(0), \gamma_n(1)$ are called the *breakpoints* of the arc spline.

Let ℓ be a line segment and $p \in \mathbb{R}^2$. A point p is *strictly left of* ℓ if $\langle \ell^\perp(0)\,,\ p - \ell(0) \rangle > 0$ and it is *strictly right of* ℓ if the inner product is negative. We say that p is *strictly left of* a positively oriented arc γ if p is in the interior of $[\gamma]$, it is *strictly left of* a negatively oriented arc γ if p is in the exterior of $[\gamma]$. Furthermore, p is left of an arc γ if it is either strictly left of γ or $p \in [\gamma]$. With $p \in \gamma^\circ$, a set $M \subset \mathbb{R}^2$ is said to be *locally left of* γ *at* p if there is an $\varepsilon > 0$ so that for every $\delta \in (0, \varepsilon)$ the set $M \cap B_p(\delta)$, with $B_p(\delta) := \{x \in \mathbb{R}^2 :\ ||x - p||_2 < \delta\}$, is nonempty and every $q \in M \cap B_p(\delta)$ is left of γ. We say that M is *locally left of* γ if for every $p \in \gamma^\circ$ it is locally left of γ at p.

Let γ be an arc, α a path and $t \in [0,1]$ with $\alpha(t) \in \gamma^\circ$. We say α *leaves* γ *in t to the left* if $\alpha(t + \varepsilon)$ is strictly left of γ for every sufficiently small $\varepsilon > 0$. Likewise, we say α *approaches* γ *in t from the left* if $\alpha(t - \varepsilon)$ is strictly left of γ for every sufficiently small $\varepsilon > 0$. Likewise, the definitions hold for "right" instead of "left".

We say that α *cuts* γ *in t from the left* if there is a $t' \in [0, t]$ with $\alpha([t', t]) \in \gamma$ such that α approaches γ in t' from the left and it leaves γ in t to the right. Likewise, we define a *cut from the right*. Note that if α is an arc then α cuts γ in t from the