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Implicitizing ruled translational surfaces ☆

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ABSTRACT

A ruled translational surface is a rational tensor product surface generated by translating a rational space curve along a straight line or equivalently translating a straight line along a rational space curve. We show how to compute the implicit equation of a ruled translational surface from two linearly independent vectors that are perpendicular to the generating line of the surface.

1. Introduction

A *ruled surface* is a surface for which through every point on the surface, there is at least one straight line that also lies on the surface. A ruled surface may also be thought of as a surface that can be swept out by moving a line in space. Ruled surfaces have been investigated extensively in classical geometry (Beauville, 1983; Edge, 1931). Due to their simplicity of form and ease of generation, ruled surfaces occur in a variety of applications including Computer Aided Design (Ravani and Chen, 1986; Ravani and Wang, 1991), architectural design (Eigensatz et al., 2010; Flöry and Pottmann, 2010), and manufacturing (Sprott and Ravani, 2008; Stute et al., 1979). Much research has been done on the computational aspects of ruled surfaces; for example, finding μ-bases for ruled surfaces (Chen et al., 2001; Chen and Wang, 2003), detecting the singularities of ruled surfaces (Izumiya and Takeuchi, 2001; Jia et al., 2009), and computing inversion formulas and reparametrizations for ruled surfaces (Busé et al., 2009; Chen, 2003).

A *translational surface* is a rational tensor product surface generated from two rational space curves by translating either one of these curves parallel to itself in such a way that each of its points describes a curve that is a translation along the other curve. Translational surfaces are basic modeling surfaces that are widely used in computer aided geometric design and geometric modeling (Pérez-Díaz and Shen, 2014; Vršek and Lávička, 2016). Since translational surfaces are generated from two rational space curves, translational surfaces have simple representations. The simplest and perhaps the most common representation of a translational surface is given by the rational parametric representation $\mathbf{h}^*(s;t) = \mathbf{f}^*(s) + \mathbf{g}^*(t)$, where $\mathbf{f}^*(s)$ and $\mathbf{g}^*(t)$ are two rational space curves.

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Recently, Vršek and Lávička (2016) and Wang and Goldman (2017) offer an alternative definition of translational surfaces given by the rational parametric representation $\mathbf{h}^*(s;t) = \frac{\mathbf{f}^*(s)+\mathbf{g}^*(t)}{2}$, where $\mathbf{f}^*(s)$ and $\mathbf{g}^*(t)$ are two rational space curves. These translational surfaces consist of all the midpoints of all the lines joining a point on $\mathbf{f}^*$ to a point on $\mathbf{g}^*$, so these translational surfaces are invariant under rigid motions: translating and rotating the two generating curves translates and rotates these translational surfaces by the same amount. Hence, applying a rigid motion to a translational surface can be achieved by applying the same rigid motion to the two rational space curves that generate the surface. Therefore, one can control these translational surfaces simply by manipulating their two generating curves. Wang and Goldman (2017) utilize syzygies to study translational surfaces. In particular, Wang and Goldman (2017) construct three special syzygies for a translational surface from a μ-basis of one of the generating space curves, show how to compute the implicit equation of a translational surface from these three special syzygies, and observe that these techniques can be applied with only minor modifications to the translational surfaces defined by $\mathbf{h}^*(s;t) = a\mathbf{f}^*(s) + b\mathbf{g}^*(t)$, where $a, b \in \mathbb{R}$ and $ab \neq 0$.

A ruled translational surface is a translational surface generated by a straight line and a rational space curve. Ruled translational surfaces are a very special type of rational surface that possess the characteristics of both ruled surfaces and translational surfaces. Hence, many computational issues concerning ruled translational surfaces can be simplified by using the properties of both ruled surfaces and translational surfaces. For example, based on the method of detecting singular points on a rational ruled surface provided by Jia et al. (2009, Theorem 3.1), it is easy to observe that the singular loci of a ruled translational surface are a collection of lines where each line passes through a singular point of the base curve $\mathbf{g}$ in the direction of the line $\mathbf{f}$.

The goal of this short paper is to compute the implicit equation of an arbitrary ruled translational surface. Thus this paper is a sequel to Wang and Goldman (2017). It is known that the resultant of a μ-basis for a rational ruled surface gives the implicit equation of the rational ruled surface (Chen et al., 2001). However, using this method, one must first compute a μ-basis. Our approach here is to employ the matrix representation of a translational surface, and use only two linearly independent vectors that are perpendicular to the generating line of the surface to compute the implicit equation of a ruled translational surface.

The method proposed in Wang and Goldman (2017) for computing the implicit equation of a translational surface from three special syzygies constructed from a μ-basis of one of the generating space curves can be applied to a ruled translational surface. Since ruled translational surfaces are a very special kind of ruled surface, as well as a particular type of translational surfaces, ruled translational surfaces possess unique properties that neither arbitrary ruled surfaces nor arbitrary translational surfaces have. Implicitization for ruled surfaces is well investigated; in particular, the algorithm for computing the implicit equation of a ruled surface from the resultant of a μ-basis proposed by Chen et al. (2001) is very efficient. This short paper presents a simple and efficient implicitization method which is geared only towards ruled translational surfaces. This result cannot be derived directly from the implicitization method for translational surfaces proposed in Wang and Goldman (2017). Furthermore, we compare the required computational steps and their complexities of our implicitization method against the implicitization method for arbitrary ruled surfaces given in Chen et al. (2001). We present a table comparing these speeds to illustrate that our method is faster.

This paper is organized in the following fashion: we start with a brief review of the definition, matrix representation, and syzygies of translational surfaces. Then we show how to compute the implicit equation of a ruled translational surface from two linearly independent vectors that are perpendicular to the generating line of the surface. We conclude with two examples and comparisons using these examples between the speed of our implicitization method and the speed of a fast implicitization method for arbitrary ruled surfaces.

2. Definitions and notation

We begin with a brief review of the basic definition of translational surfaces, and the syzygy related properties of translational surfaces. For additional details and results concerning syzygies and μ-bases for translational surfaces, see Wang and Goldman (2017).

A *translational surface* is a rational tensor product surface $\mathbf{h} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$:

$$\mathbf{h}(s,u;t,v) = [h_0(s,u;t,v), \ldots, h_3(s,u;t,v)] = f_0(s,u)\mathbf{g}(t,v) + g_0(t,v)\mathbf{f}(s,u) \tag{1}$$

generated by two rational space curves $\mathbf{f}(s,u)$ and $\mathbf{g}(t,v)$, where $\mathbf{f}, \mathbf{g} : \mathbb{P}^1 \to \mathbb{P}^3$,

$$\mathbf{f}(s,u) = [f_0(s,u),\ f_1(s,u),\ f_2(s,u),\ f_3(s,u)], \quad \mathbf{g}(t,v) = [g_0(t,v),\ g_1(t,v),\ g_2(t,v),\ g_3(t,v)],$$

f_i and g_i are homogeneous polynomials with homogenizing parameters u and v in the standard $\mathbb{Z}$-graded rings $\mathbb{R}[s,u]$ and $\mathbb{R}[t,v]$ of $\deg(f_i) = m$, $\deg(g_i) = n$, and $\gcd(f_i) = \gcd(g_i) = 1$ for $i = 1,2,3,4$. Note that the functions h_i are bihomogeneous polynomials in the bigraded ring $\mathbb{R}[s,u;t,v]$ of bidegree $\deg(h_i) = (m,n)$, where $\deg(s) = \deg(u) = (1,0)$ and $\deg(t) = \deg(v) = (0,1)$. From now on, we shall write $\mathbf{h}(s,u;t,v)$ as the bihomogeneous parametrization of the translational surfaces given by Equation (1). For definitions and analysis of alternative types of translational surfaces, see Wang and Goldman (2017).

A *ruled translational surface* is a translational surface of bidegree $(m,1)$ or $(1,n)$. Without loss of generality, throughout this paper, we assume that $m = 1$, that is, the curve $\mathbf{f}(s,u)$ is a straight line.

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