Computer Aided Geometric Design 57 (2017) 23–43

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

Mesh-based and meshless design and approximation of scalar functions

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ARTICLE INFO

Article history:
Available online 31 May 2017

Keywords:
Scalar function design
Meshless approximation
Critical points
Laplacian matrix
Visualisation
Applications

ABSTRACT

In engineering, geographical applications, bio-informatics, and scientific visualisation, a variety of phenomena is described by data modelled as the values of a scalar function defined on a surface or a volume, and critical points (i.e., maxima, minima, saddles) usually represent a relevant information about the input data or an underlying phenomenon. Furthermore, the distribution of the critical points is crucial for geometry processing and shape analysis; e.g., for controlling the number of patches in quadrilateral remeshing and the number of nodes of Reeb graphs and Morse–Smale complexes. In this context, we address the design of a smooth function, whose maxima, minima, and saddles are selected by the user or imported from a template (e.g., Laplacian eigenfunctions, diffusion maps). In this way, we support the selection of the saddles of the resulting function and not only its extrema, which is one of the main limitations of previous work. Then, we discuss the meshless approximation of an input scalar function by preserving its persistent critical points and its local behaviour, as encoded by the spatial distribution and shape of the level-sets. Both problems are addressed by computing an implicit approximation with radial basis functions, which is independent of the discretisation of differential operators and of assumptions on the sampling of the input domain. This approximation allows us to introduce a meshless iso-contouring and classification of the critical points, which are characterised in terms of the differential properties of the meshless approximation and of the geometry of the input surface, as encoded by its first and second fundamental form. Furthermore, the computation is performed at an arbitrary resolution by locally refining the input surface and by applying differential calculus to the meshless approximation. As main applications, we consider the approximation and analysis of scalar functions on both 3D shapes and volumes in graphics, Geographic Information Systems, medicine, and bio-informatics.

1. Introduction

In several applications (e.g., engineering, geographical applications, bio-informatics, and scientific visualisation), scalar functions defined on a surface or a volume are used to model a relevant information about the input data or an underlying phenomenon. Controlling the distribution of the critical points (i.e., maxima, minima, saddles) of a scalar function during its design is also crucial for geometry processing and shape analysis. For instance, properly designing a scalar function with

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http://dx.doi.org/10.1016/j.cagd.2017.05.005
0167-8396/

a prescribed set of critical points provides a flexible control over the number, shape, and size of the resulting quadrangular patches of remeshed surfaces (Dong et al., 2006; Huang et al., 2008; Ni et al., 2004), the number of nodes of the Reeb graph (Pascucci et al., 2007; Patanè et al., 2009) and of the Morse–Smale complex. However, harmonic functions allow the user to select only the number and position of its maxima and minima, with no control on the corresponding saddles. In a similar way, Laplacian eigenfunctions associated with small eigenvalues and diffusion maps at small scales are characterised by a generally low number of critical points, whose spatial location cannot be defined *a-priori*.

After a brief review of previous work (Sect. 2), our first goal (Sect. 3) is to design a smooth function whose maxima, minima, and saddles are selected by the user or imported from a template function. As main examples of template functions, we mention the Laplacian eigenfunctions (Reuter et al., 2009) and the diffusion maps (Bronstein and Bronstein, 2011; Patanè, 2017), which are intrinsically defined by the input surface and their critical points represent relevant shape features, such as protrusions and high-curvature regions. The function with designed critical points is computed by applying either a mesh-based or a meshless approximation and by combining interpolating or least-squares constraints with the spectral properties of the Laplacian matrix. Then, the resulting problem reduces to the solution of a linear system. As novel contribution with respect to previous work, we allow the user to select the saddles of the designed function, and not only its extrema.

Our second goal (Sect. 4) is to compute a meshless approximation of an input scalar function by preserving its persistent critical points and its local behaviour (e.g., as encoded by the level-sets), instead of minimising only the approximation error as done by previous work. The proposed approximation preserves the topology of the *global structure* of the input scalar function by applying interpolating constraints on its values at the persistent critical points and at their 1-star vertices. It also approximates the *local details* according to the target accuracy, through least-squares constraints on the function values at a sub-sampling of the level-sets. The resulting function provides a good approximation accuracy without over-fitting the input data, and is robust against noise and local perturbations. As novelty with respect to previous work, we focus on the preservation of the spatial distribution and shape of the level-sets, which are useful to characterise the function behaviour. For instance, in medicine the shape of the level-sets is useful to discriminate between pathological and healthy cases, in Geographic Information Systems (GIS) it characterises the terrain morphology, and in Computer Graphics it encodes geometric properties of shape-driven functions, such as geodesic and diffusion distances.

The meshless techniques underlying the design and approximation of scalar functions with constrained critical points support the computation of the level-sets, the classification of the critical points, and the encoding of the approximations in a compact representation that saves input/output space (Sect. 5). All these computations are performed at an arbitrary resolution by locally refining the input surface and by applying differential calculus to the meshless approximation.

2. Previous work

We briefly summarise previous work on the classification and simplification of the critical points, and on the approximation of discrete data with implicit methods and radial basis functions.

Critical points. Given a $\mathcal{C}^1$ function $f:\mathcal{M}\rightarrow\mathbb{R}$ defined on a smooth 2-manifold surface $\mathcal{M}$, the *critical points* of f are defined as those points $\mathbf{p}\in\mathcal{M}$ such that $\nabla f(\mathbf{p})=\mathbf{0}$ and they correspond to the maxima, minima, and saddles of f. For polyhedral surfaces, the method described in Banchoff (1967) classifies a vertex according to the values of f on its neighbourhood. If $\mathcal{M}$ is a triangle mesh, then the vertex $\mathbf{p}$ is a *maximum* or a *minimum* if its function value is higher or lower than those ones on its 1-*star*, respectively. We briefly remind that the 1-star of a vertex $\mathbf{p}_i$ is defined as the set of vertices incident to $\mathbf{p}_i$; i.e., $\{\mathbf{p}_j:\ (\mathbf{p}_i,\mathbf{p}_j)\ \text{edge}\}$. If two or more iso-curves related to the same iso-value share a vertex $\mathbf{p}$, then $\mathbf{p}$ is a *saddle*. Those points that do not fall in the previous classification are defined as *regular*. According to this classification, the *Euler formula* $\chi(\mathcal{M})=m-s+M$ provides a link between the number of critical points (i.e., m minima, M maxima, s saddles) of the input scalar function and the Euler characteristic $\chi(\mathcal{M})$ of $\mathcal{M}$. For details on the classification of critical points for piecewise linear scalar functions on triangle meshes, we refer the reader to Bremer et al. (2004) and Sect. 3.1.

Topological simplification of critical points based on persistence. Given a scalar function $f:\mathcal{M}\rightarrow\mathbb{R}$ with a large number of critical points associated with a low variation of the corresponding f-values, previous work (Bremer et al., 2004; Edelsbrunner et al., 2004) defines a topological hierarchy for f that is constructed by performing a progressive simplification of the Morse complex $\mathcal{F}$ of f through the cancellation of pairs of critical points. Then, the critical points are paired by visiting $\mathcal{M}$ with respect to the reordering of its vertices according to increasing values of f. The importance weight associated with the pair $(\mathbf{p}_i,\mathbf{p}_j)$ is measured as the *persistence* of $\mathbf{p}_i$, $\mathbf{p}_j$, that is, $|f(\mathbf{p}_i)-f(\mathbf{p}_j)|$. The local updates of the complex are performed by iteratively removing those pairs with the lowest persistence and reconnecting the neighbours of the removed nodes. Each node removal affects the number and configuration of the critical points of $\mathcal{F}$ without changing f. Therefore, the simplification provides a hierarchy for f where each Morse complex $\mathcal{F}^{(k)}$ is not associated with a corresponding scalar function $f^{(k)}$ on $\mathcal{M}$.

In Edelsbrunner et al. (2006), the input scalar function f is replaced with a new function $\tilde{f}$ that has the same points of persistence of f higher than a given threshold ε and the $\mathcal{L}_\infty$-error between f and $\tilde{f}$ is lower than ε. The ε-*simplification* of the structure of f and the construction of $\tilde{f}$ are based on an iterative process, which cancels minimum-saddle pairs by sweeping the vertices from bottom to top and lower the saddles that belong to a pair of persistence lower than ε.