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Affine geometry of equal-volume polygons in 3-space ☆

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To the memory of Gerald Farin.

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ABSTRACT

Equal-volume polygons are obtained from adequate discretizations of curves in 3-space, contained or not in surfaces. In this paper we explore the similarities of these polygons with the affine arc-length parameterized smooth curves to develop a theory of discrete affine invariants. Besides obtaining discrete affine invariants, equal-volume polygons can also be used to estimate projective invariants of a planar curve. This theory has many potential applications, among them evaluation of the quality and computation of affine invariants of silhouette curves.

1. Introduction

Affine differential geometry studies differential concepts that are invariant under the action of the affine group. In this paper we shall discuss affine differential invariants of curves in 3-space. We shall be particularly interested in curves contained in surfaces and curves together with a distinguished origin, which is the object of centro-affine differential geometry. We shall also consider some results of planar projective geometry, which studies projective invariants of curves.

Discrete differential geometry is a quickly developing field whose aim is to obtain discrete counterparts of the differential concepts keeping their main properties not only approximately, but exactly. It is a common belief that such discrete counterparts also presents a better numerical behavior (Bobenko and Suris, 2008). In this paper we shall propose discrete counterparts of the affine differential concepts of curves in 3-space that keep many of the corresponding smooth properties, and so these objects can be considered as part of discrete differential geometry.

We say that a smooth curve $\gamma(t)$ in 2-space is parameterized by *affine arc-length* if $[\gamma'(t), \gamma''(t)] = 1$, where $[\cdot,\cdot]$ denotes the determinant of 2 vectors (Buchin, 1983; Izumiya and Sano, 1998a). For polygons, the corresponding condition is that the area of the triangle determined by three consecutive vertices is constant. Planar polygons satisfying this condition are called *equal-area* and the affine geometry of these polygons has been recently studied (Craizer et al., 2012; Käferböck, 2014). In this paper we generalize this study to polygons in 3-space by considering the concept of equal-volume polygons.

For a smooth curve ϕ contained in a surface M, we say that the parameterization $\phi(t)$ is *adapted to* M if

$$[\phi'(t), \phi''(t), \xi(t)] = 1, \tag{1.1}$$

where $[\cdot,\cdot,\cdot]$ denotes the determinant of 3 vectors and ξ is the parallel Darboux vector field of $\phi \subset M$ (Craizer et al., 2016). In centro-affine geometry, we consider curves ϕ in 3-space together with a distinguished origin O, and we say that $\phi(t)$ is parameterized by centro-affine arc-length with respect to O if

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$$[\phi(t) - O, \phi'(t), \phi''(t)] = 1, \tag{1.2}$$

(Giblin and Sano, 2012). A smooth curve $\Phi(t)$ in 3-space is parameterized by *affine arc-length* if

$$[\Phi'(t), \Phi''(t), \Phi'''(t)] = 1, \tag{1.3}$$

(Davis, 2006; Izumiya and Sano, 1998b). We observe that, in all these contexts, the basic condition is the constancy of some volume. In this paper, we describe polygons in 3-space whose corresponding volumes are constant, and we call them *equal-volume* polygons. We obtain affine invariant measures only for these equal-volume polygons, but we also describe a simple algorithm that, by re-sampling an arbitrary polygon, obtain an equal-volume one.

We say that a smooth curve ϕ contained in a surface M is *non-degenerate* if its osculating plane does not coincide with the tangent plane of M at any point. For such curves, there exists a vector field ξ tangent to M and transversal to ϕ such that its derivative ξ' is tangent to ϕ, i.e.,

$$\xi' = -\sigma\phi', \tag{1.4}$$

for some scalar function σ. This vector field is unique up to a multiplicative constant and is called the *parallel Darboux vector field*. It turns out that $\phi \subset M$ is a silhouette curve with respect to some point O if and only if σ is constant (Craizer et al., 2016; Izumiya and Otani, 2015). As a discrete model for curves contained in surfaces, consider a polyhedron M whose faces are planar quadrilaterals and let $\phi(i)$ be vertices of a polygon ϕ whose sides are connecting opposite edges of a face of M. We can choose a vector field $\xi(i)$ in the direction of the edges of M containing the vertices $\phi(i)$ such that the difference $\xi(i+1) - \xi(i)$ is parallel to the corresponding side of the polygon ϕ. Then equation (1.4) still holds, for some scalar function σ, replacing derivatives by differences. We prove that ϕ is a silhouette polygon for the polyhedron M if and only if σ is constant. This result may be used as a measure of quality of a silhouette polygon.

A non-degenerate curve $\phi \subset M$ admits a parameterization satisfying equation (1.1), unique up to a translation. For silhouette curves relative to O, taking $\sigma = -1$ and $\xi = \phi - O$, equation (1.1) reduces to equation (1.2). In section 3 we shall define a discrete counterpart of equation (1.1) and say that a polygon ϕ contained in a polyhedron M is *equal-volume* if it satisfies this equation. For silhouette polygons ϕ relative to O, we may take $\sigma = -1$ and $\xi = \phi - O$. In this case the equal-volume condition becomes a discrete counterpart of equation (1.2).

The plane $\mathcal{A} = \mathcal{A}(t)$ generated by $\{\phi(t), \phi''(t)\}$ is called the *affine normal plane*, while the envelope $\mathcal{B}$ of these affine normal planes is a developable surface $\mathcal{B}$ called the *affine focal set* of the pair $\phi \subset M$ (Craizer et al., 2016, 2017). The smooth curves $\phi \subset M$ whose affine focal set $\mathcal{B}$ reduces to a single line were characterized in Craizer et al. (2017). For equal-volume polygons, define the *affine normal plane* $\mathcal{A}(i)$ as the plane generated by $\{\phi(i), \phi''(i)\}$ and the *affine focal set* $\mathcal{B} = \mathcal{B}(\phi, M)$ as a discrete envelope of these affine normal planes. We characterize in this paper those equal-volume polygons contained in a polyhedron whose affine focal set reduces to a single line.

Consider a smooth curve $\Phi(t)$ in 3-space parameterized by affine arc-length, i.e., satisfying equation (1.3). The planes through Φ parallel to $\{\Phi', \Phi'''\}$ are called *affine rectifying planes* and the envelope of the affine rectifying planes $RS(\Phi)$ is called the *intrinsic affine binormal developable* (Izumiya and Sano, 1998b). The characterization of curves Φ such that $RS(\Phi)$ is cylindrical is easily obtained from the characterization of curves $\phi = \Phi'$ whose affine focal set is a single line (Craizer et al., 2017; Izumiya and Sano, 1998b). A polygon $\Phi(i + \frac{1}{2})$ in 3-space is said to be *equal-volume* if the difference polygon $\phi(i) = \Phi'(i)$ is equal-volume with respect to the origin. Although it is not clear how to obtain a discrete version of the intrinsic affine binormal developable, we can obtain interesting consequences of the discrete characterization of polygons ϕ whose affine focal set is a single line.

We can also apply the equal-volume model in a projective setting. Given a smooth planar curve $(\tilde{\phi}(t), 1)$, there exists a projectively equivalent curve $\phi(t)$ in 3-space satisfying equation (1.2) with O equal the origin. From this curve, we can define the projective length $pl(\tilde{\phi})$ (see Guieu et al., 1997). For a planar polygon $(\tilde{\phi}(i), 1)$, we can also obtain a projectively equivalent equal-volume polygon $\phi(i)$ in 3-space and, from this polygon, we obtain two definitions for the projective length, $pl_1(\tilde{\phi})$ and $pl_2(\tilde{\phi})$, that unfortunately do not coincide. Nevertheless, we prove that if the polygon is obtained from a dense enough sampling of a smooth curve, both the discrete projective length $pl_1(\tilde{\phi})$ and $pl_2(\tilde{\phi})$ are close to the projective length of the smooth curve.

The paper is organized as follows: In section 2 we review the smooth results of affine geometry of curves contained in surfaces, centro-affine geometry of curves in 3-space, affine geometry of curves in 3-space and projective geometry of planar curves. Section 3 is the main section of the paper, where we calculate affine invariants of equal-volume polygons contained in polyhedra. In section 4, we apply the results of section 3 to compute affine invariants for equal-volume polygons in 3-space. In section 5 we discuss the projective length of a planar polygon.

2. Affine geometry of smooth curves in 3-space

In this section we present without proofs some results of affine differential geometry of smooth curves in 3-space. In the following sections, we shall describe and prove discrete counterparts of these results.