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A fully data-dependent criterion for free angles selection in spatial PH cubic biarc Hermite interpolation

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ABSTRACT

Cubic biarcs are the natural counterpart, in the context of the PH spatial Hermite interpolation, of piecewise quadratics for Hermite interpolation of standard parametric curves. Recently, it has been shown that, spatial Hermite interpolation can be always treated with PH cubic biarcs, providing an interesting alternative to the use of higher degrees, see Bastl et al. (2014). Beside the parameter value at the joint of the two arcs, which is typically free in any scheme of this kind, two free angular parameters remain. They should be automatically fixed by some suitable criterion in order to ensure the generation of interpolants with a reasonable shape. In this paper, an alternative to the constant choice of free parameters presented in Bastl et al. (2014) is proposed. It consists of an extension of the so called CC selection strategy introduced in Farouki et al. (2008) to the biarc setting. Such strategy is fully data-dependent, does not require any special configuration of the coordinate system and it guarantees the PH cubic reproduction when such kind of the interpolant exists. Moreover, the obtained scheme possesses other two important features, i.e. it is third order accurate and gives the possibility to control the torsion sign of the produced interpolant, in case with the introduction of an additional tension parameter relaxing the smoothness from C^1 to G^1. The numerical results, based also on the straightforward spline extension of the scheme, confirm the developed theoretical analysis.

1. Introduction

A Pythagorean–Hodograph (PH) curve is a parametric (piecewise) polynomial curve characterized by the fact that its parametric speed is (*piecewise*) *polynomial* in the curve parameter. As a consequence of this distinctive feature, PH curves possess important properties which are particularly useful in many applications ranging from Computer Numerical Control machining to the construction of rigid body motions. For a brief review of the construction and the properties of PH curves, see Farouki (2008).

The Bézier control points of a PH curve are nonlinearly constrained. Therefore, the development of the related geometric algorithms is fundamental for their practical use, and C^1 or G^1 Hermite interpolation is often considered because it can be achieved with local schemes based on low degree PH curves. Usually, to address the classical first order Hermite

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interpolation problem for parametric curves in $\mathbb{R}^d$, cubic curves or piecewise quadratic curves, with a break point $\hat{t}$ in the parametric data interval, are used. In both cases the Hermite conditions are necessary and sufficient to fix the $4d$ free scalar parameters defining the curve. This is not the case for the PH curves which need a suitable degree raising.

For planar Hermite data the problem has been firstly addressed in Farouki and Neff (1995) by using PH quintics and it is shown that four distinct PH interpolants can be defined, see also Lee et al. (2012) for a recent interesting rational alternative based on the use of Möbius transformations of PH cubics. In the space the problem can be treated with PH quintics as well and in this case the selection of good interpolants becomes more challenging because there exists a two-parameter family of solutions, as first shown in Farouki et al. (2002). Among the other criteria for the selection of the free parameters, we mention the so called CC (Cubic Cubic) criterion firstly introduced in Farouki et al. (2008) and later better analyzed in Sestini et al. (2013) and in Farouki et al. (in press).

In the present paper by using the quaternion representation of PH curves, see Choi et al. (2002), the problem of spatial C^1 Hermite interpolation based on PH cubic biarcs is studied. In addition to the reduced degree, the cubic biarcs approach is of sure interest in the context of modeling because the composite Bézier control polygon mimics quite precisely the constructed curve. In particular, it is tangent to the curve at the point where the two arcs join. On the other hand, the control polygon of a quintic curve gives in general a quite rough idea of the underlying curve.

This problem was firstly addressed in Bastl et al. (2014) where it was shown that it results in a three-parameter family of solutions, namely one free parameter corresponds to the parameter $\hat{t}$ where the two arcs join and the other two are angular parameters. On this concern it was already outlined in the mentioned paper that a technique for an automatic selection of the angles is useful in practice since arbitrary choices can produce very unnatural interpolants. In Bastl et al. (2014), a scheme based on an asymptotic analysis is proposed, more precisely, to get the maximal approximation order, i.e. 3, both angles are set to zero under the necessary assumption of alignment of the bisector between the two end unit tangents and the vector used to represent the hodograph of the PH biarc. Note that the same general approach was adopted also in Farouki et al. (2002) and in Šír and Jüttler (2005).

In this paper, a different strategy, extending the CC technique introduced in Farouki et al. (2008) to the biarc context, is presented. It does not require any special data configuration and guarantees the reproduction of the PH cubic interpolant when it exists. Furthermore, it possesses other two important features which are proved and numerically tested in the paper. First, the obtained interpolant has the maximal approximation order 3 for any admissible choice of the free parameter $\hat{t}$. Second, it has the capability of preserving the sign of the discrete torsion of the Hermite data, such preservation possibly requiring to reduce the smoothness at the end points from C^1 to G^1. In fact, following the approach considered in Farouki et al. (in press) assuming that the lengths of the two end derivatives can be multiplied by a free positive tension parameter $\sigma^2 \le 1$, we prove that, if σ is sufficiently small, the sign of the torsion of the cubic PH biarc interpolant matches with that of the discrete torsion of the Hermite data. Note that in practice the tension parameter σ is initialized to 1 and it is reduced only if it is necessary to get the desired torsion control. Finally, we emphasize that in case $\hat{t} = 0.5$ there is a numerical evidence that the two PH cubic arcs obtained by the CC technique are also torsion continuous, making this choice very appealing.

Some numerical experiments that demonstrate the important features of our scheme are presented and the comparison with the interpolants obtained by using the angle selection proposed in Bastl et al. (2014) is given.

The paper assumes that the reader is familiar with the quaternion algebra (in case a short introduction can be found for example in Section 2 of Farouki et al., 2008) and it is organized as follows. To make the paper self-contained and to introduce the necessary notation, Section 2 is devoted to the presentation of the interpolation problem, although it was already studied in Bastl et al. (2014). In Section 3 the technique introduced in Farouki et al. (2008) is extended to the biarc context in order to determine two free angular parameters. The approximation order of the obtained scheme is studied in Section 4, and in Section 5 it is shown how the use of an auxiliary tension parameter can ensure that our interpolant preserves the sign of the discrete torsion of the data. Finally, Section 6 presents some numerical results also obtained by using the straightforward spline implementation of the scheme. The main results of the paper are summarized in the Conclusion section.

2. The problem

A general n degree biarc $\mathbf{r} : [0, 1] \mapsto \mathbb{R}^3$ can be represented in Bernstein form as follows,

$$\mathbf{r}(t) = \begin{cases} \mathbf{a}(t) := \sum_{i=0}^{n} \mathbf{a}_i B_i^n\left(\frac{t}{\hat{t}}\right), & \text{if } t \in [0, \hat{t}], \\ \mathbf{b}(t) := \sum_{i=0}^{n} \mathbf{b}_i B_i^n\left(\frac{t-\hat{t}}{1-\hat{t}}\right), & \text{if } t \in [\hat{t}, 1], \end{cases} \tag{1}$$

where B_i^n denotes the i-th element of the Bernstein basis of polynomials of degree less or equal to n and $\mathbf{a}_i, \mathbf{b}_i$, $i = 0, \dots, n$, are the Bézier control points characterizing two polynomial segments of $\mathbf{r}$, where at $t = \hat{t}$ at least C^0 continuity is required. In general the value $\hat{t} \in (0, 1)$ where the two polynomial segments join is a free shape parameter.