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Transition to canonical principal parameters on minimal surfaces

Ognian Kassabov

University of Transport, 158 Geo Milev Str., 1574 Sofia, Bulgaria

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ABSTRACT

Ganchev has recently proposed a new approach to minimal surfaces. Introducing canonical principal parameters for these surfaces, he has proved that the normal curvature determines the surface up to its position in the space. Here we prove a theorem that permits to obtain equations of a minimal surface in canonical principal parameters and we make some applications on parametric polynomial minimal surfaces. Thus we show that Ganchev's approach implies an effective method to prove the coincidence of two minimal surfaces given in isothermal coordinates by different parametric equations.

1. Introduction

In the differential geometry of space curves the natural parameters are an important tool. These parameters are used to define the natural equations of the space curves, the curvature and the torsion, which are objects that determine the curves up to position in the space. In particular, a plane curve is determined up to position in the plane by its curvature.

In the geometry of surfaces we haven't been aware of analogous natural parameters and natural equations until now. Recently natural parameters were introduced for a wide class of surfaces, the so-called Weingarten surfaces, see Ganchev and Mihova (2010). Some results of this article were specialized in Ganchev (2008) for the class of minimal surfaces and canonical principal parameters were introduced. Like the natural parameters of the space curves they are determined up to a sign and additive constants. Moreover, the normal curvature of a minimal surface expressed in these parameters determines completely the surface up to a position in the space – just like the curvature of a plane curve.

Note that a surface may appear in quite different parametrizations and it is not easy to say whether or not two equations define the same surface. For example the catenoid may be represented in nonparametric form, i.e. as a graph of a function

$$z = \operatorname{arccosh}\sqrt{x^2 + y^2}$$

or with any of the following parametric equations:

$$\mathbf{x}(u, v) = \left(\frac{u}{2}\left(1 + \frac{1}{u^2 + v^2}\right), \frac{v}{2}\left(1 + \frac{1}{u^2 + v^2}\right), \frac{1}{2}\ln\left(u^2 + v^2\right)\right),$$

$$\tilde{\mathbf{x}}(u, v) = (\cosh u \cos v, \cosh u \sin v, u).$$

To prove that the surfaces defined by two equations coincide we can look for a change of the parameters and/or the coordinate system, or use a theorem from Eisenhart (1909). But any one of these methods is an arduous task in general. The

E-mail address: okassabov@abv.bg.

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results of Ganchev imply a very useful method in this direction for minimal surfaces: we may simply calculate the normal curvature of these surfaces in canonical principal parameters and then compare the results. The problem is that usually a minimal surface is defined in arbitrary isothermal parameters.

Here we find a differential equation that gives us a possibility to make the transition from isothermal parameters of a minimal surface to canonical principal parameters. Then we make some applications of this result. We find the group of transformations on a holomorphic function that preserves the minimal surface generated by this function, so we join any minimal surface with a class of holomorphic functions.

Our results will be useful in the CAD research. Indeed one of the main tasks in the computer aided geometric design is to find a surface with prescribed border that minimizes the area among all the surfaces with the same border. A classical result says that such a surface is minimal. Because of the high nonlinearity of the area functional and the development of the theory of Bézier surfaces it is more convenient to consider a restricted problem: among all parametric *polynomial* surfaces with prescribed border find a surface with a minimal area, see e.g. Monterde (2004). A similar problem for biharmonic Bézier surfaces is considered in Monterde and Ugail (2004). Some of the results of Monterde (2004) and Monterde and Ugail (2004) are generalized in Wu and Wang (2012). In Xu et al. (2009) is proposed the use of a different functional for minimization (it is expressed by the mean and the Gauss curvature) and an optimization method is presented for Bézier surfaces.

Of course, because of the advantages of the minimal surfaces, it is best to consider a non-restricted problem and to obtain if possible a polynomial minimal surface as a result. This leads to the questions: which parametric polynomial surfaces are minimal and how they differ. A possible way to answer the second question is to use the method given by the results in Sections 3 and 4 of this paper.

It is proved in Cosín and Monterde (2002) that the parametric polynomial minimal surfaces of degree 3 in isothermal parameters are too rigid to be used in CAD works as a consequence of the fact that the Enneper surface is the only such surface. On the other hand in Xu and Wang (2010) some families of parametric minimal surfaces of degree 5 are introduced and some of their beautiful properties are obtained. Applying our method, in Section 5 we obtain a surprising result: these families actually coincide and any surface in them is homothetic to the surface generated via the Weierstrass formula with $f(z) = z^2$, $g(z) = z$. A similar result follows for some polynomial surfaces of degree 6, introduced in Xu and Wang (2008). So the structure of these families is now clear.

2. Preliminaries

Let S be a regular surface defined by the parametric equation

$$\mathbf{x} = \mathbf{x}(u, v) = \big(x_1(u, v), x_2(u, v), x_3(u, v)\big), \quad (u, v) \in U \subset \mathbb{R}^2.$$

Denote the derivatives of the vector function $\mathbf{x} = \mathbf{x}(u, v)$ by

$$\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}, \qquad \mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}.$$

The coefficients of the first fundamental form are given by

$$E = \mathbf{x}_u^2, \qquad F = \mathbf{x}_u \mathbf{x}_v, \qquad G = \mathbf{x}_v^2,$$

and the unit normal to the surface is

$$\mathbf{U} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}.$$

The second derivatives of $\mathbf{x}(u, v)$ are denoted by $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$, $\mathbf{x}_{vv}$, respectively. Then the coefficients of the second fundamental form are given by

$$L = \mathbf{U}\mathbf{x}_{uu}, \qquad M = \mathbf{U}\mathbf{x}_{uv}, \qquad N = \mathbf{U}\mathbf{x}_{vv}.$$

Suppose that the principal lines of S are parametric lines and the surface has no umbilic points. Then $F = M = 0$ and the principal curvatures ν_1 and ν_2 are expressed by

$$\nu_1 = \frac{L}{E}, \qquad \nu_2 = \frac{N}{G}.$$

The Gauss curvature K and the mean curvature H of a surface S are defined by

$$K = \frac{LN - M^2}{EG - F^2} = \nu_1 \nu_2, \qquad H = \frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{\nu_1 + \nu_2}{2}.$$

Note that the Gauss curvature of a surface does not depend on the parametrization while the mean curvature is an invariant up to a sign. The surface S is said to be *minimal* if its mean curvature vanishes identically. In this case the Gauss curvature