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A Hermite interpolatory subdivision scheme for C^2-quintics on the Powell–Sabin 12-split

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ABSTRACT

In order to construct a C^1-quadratic spline over an arbitrary triangulation, one can split each triangle into 12 subtriangles, resulting in a finer triangulation known as the Powell–Sabin 12-split. It has been shown previously that the corresponding spline surface can be plotted quickly by means of a Hermite subdivision scheme (Dyn and Lyche, 1998). In this paper we introduce a nodal macro-element on the 12-split for the space of quintic splines that are locally C^3 and globally C^2. For quickly evaluating any such spline, a Hermite subdivision scheme is derived, implemented, and tested in the computer algebra system `Sage`. Using the available first derivatives for Phong shading, visually appealing plots can be generated after just a couple of refinements.

1. Introduction

For approximating functions on a given domain, a popular method is to triangulate the domain and consider an approximation in a space $\mathcal{S}$ of piece-wise polynomials over the triangulation. It is a hard problem to find a basis $\mathcal{B}$ of $\mathcal{S}$ that has all the usual properties of the univariate B-splines.

One desired property of $\mathcal{B}$ is that it is *local*, meaning that each spline in $\mathcal{B}$ has local support. One way to construct such a local basis is to first split each triangle into several subtriangles, and then construct a basis on the refined triangulation.

A popular split is the Powell–Sabin 12-split (Powell and Sabin, 1977); see Fig. 1(a) and Section 3. While the 12-split splits the triangle in a relatively large number of subtriangles, a major advantage over other well-known splits stems from the following property (Dyn and Lyche, 1998; Oswald, 1992). Let be given a triangle $\triangle$, its 12-split $\triangle$, and the split $\triangle$, where we subdivided $\triangle$ into four subtriangles by connecting the midpoints of the edges. If we replace each subtriangle in $\triangle$ by its 12-split, the space of splines over the resulting split contains the space of splines over $\triangle$. This refinability property makes the 12-split suitable for multiresolution analysis.

Recently, a simplex spline basis for the C^1-quadratics on the 12-split with all the usual properties of the univariate B-spline basis was discovered (Cohen et al., 2013). Powell and Sabin originally constructed a nodal basis (see Section 2) on the 12-split, which can be used to represent C^1-smooth quadratic splines over arbitrary triangulations. Schumaker and Sorokina viewed the space of C^1-quadratics on the 12-split as the first entry in a sequence of spline spaces of increasing smoothness and degree (Schumaker and Sorokina, 2006). The second entry is a space of C^2-quintics, with C^3-supersmoothness at the vertices and midpoints and satisfying some additional C^3-conditions of type (4) along some

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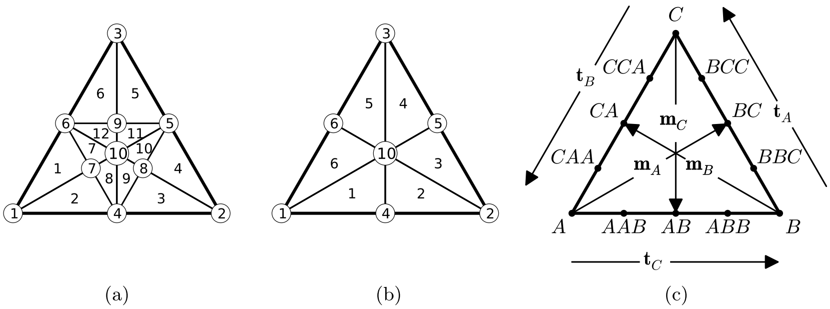

Fig. 1. The Powell–Sabin 12-split (a) and 6-split (b) with labeling of vertices and faces. (c) A triangle with corners $\mathbf{v}_A, \mathbf{v}_B, \mathbf{v}_C$, midpoints $\mathbf{v}_{AB}, \mathbf{v}_{BC}, \mathbf{v}_{CA}$, quarterpoints $\mathbf{v}_{AAB}, \mathbf{v}_{ABB}, \mathbf{v}_{BBC}, \mathbf{v}_{BCC}, \mathbf{v}_{CCA}, \mathbf{v}_{CAA}$, medial vectors $\mathbf{m}_A, \mathbf{m}_B, \mathbf{m}_C$, and tangential vectors $\mathbf{t}_A, \mathbf{t}_B, \mathbf{t}_C$.

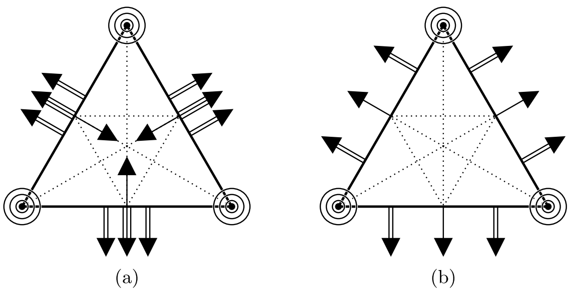

Fig. 2. Schumaker and Sorokina's macro-element (a) and a new macro-element (b) on the 12-split. A bullet represents a point evaluation, three circles represent all derivatives up to order three, and a single, double, and triple arrow represent a first-, second-, and third-order directional derivative. These derivatives are evaluated at the rear end of the arrows, which are located at midpoints and adjacent domain points (a), and at the midpoints and quarter-points (b).

of the interior edges. On a single triangle this space has dimension 42, and the authors constructed a nodal macro-element for this space; see Fig. 2(a).

For a recent and similar construction see Davydov and Yeo (2013). A family of smooth spline spaces on the 6-split and corresponding normalised bases were presented in Speleers (2013). For other refinable C^1-quadratic elements on 6-splits see Dæhlen et al. (2000); Maes and Bultheel (2006) and Jia and Liu (2008). In the latter a combination of 6- and 12-splits is used. For the FVS C^1-cubic quadrangular macro element see Davydov and Stevenson (2005); Hong and Schumaker (2004), and for a survey of refinable multivariate spline functions see Goodman and Hardin (2006).

The next section reviews some standard Bernstein–Bézier techniques. Section 3 introduces a new macro-element space for C^2-quintics, with complete C^3-smoothness within each macrotriangle and dimension only 39; see Fig. 2(b). In the following two sections a Hermite subdivision scheme is derived, implemented, and tested in the computer algebra system `Sage`. A concluding remark briefly explains why the macro-element presented in this paper admits no trivial extension to higher degree and smoothness.

2. Bernstein–Bézier techniques

We follow the notation from Lai and Schumaker (2007). Any point $\mathbf{v}$ in a nondegenerate triangle $T = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$ can be represented by its *barycentric coordinates* (b_1, b_2, b_3), which are uniquely defined by $\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3$ and $b_1 + b_2 + b_3 = 1$. Similarly, each vector $\mathbf{u}$ is uniquely described by its *directional coordinates*, i.e., the triple $(b_1 - b_1', b_2 - b_2', b_3 - b_3')$ with (b_1, b_2, b_3) and (b_1', b_2', b_3') the barycentric coordinates of two points $\mathbf{v}$ and $\mathbf{v}'$ such that $\mathbf{u} = \mathbf{v} - \mathbf{v}'$.

A polynomial p of degree d defined on T is conveniently represented by its *Bézier form*

$$p(\mathbf{v}) = \sum_{i+j+k=d} c_{ijk} B^d_{ijk}(\mathbf{v}), \qquad B^d_{ijk}(\mathbf{v}) := \frac{d!}{i!j!k!} b_1^i b_2^j b_3^k,$$

where the B^d_{ijk} are referred to as the *Bernstein basis polynomials* of degree d and the c_{ijk} are called the *B-coefficients* of p. We associate each B-coefficient c_{ijk} to the *domain point* $\xi_{ijk} := \frac{i}{d}\mathbf{v}_1 + \frac{j}{d}\mathbf{v}_2 + \frac{k}{d}\mathbf{v}_3$. The *disk of radius m around* $\mathbf{v}_1$ is $D_m(\mathbf{v}_1) := \{\xi_{ijk} : i \geq d - m\}$, and similarly for the other vertices.

For any differentiable function $f : \Omega \longrightarrow \mathbb{R}$ and a vector $\mathbf{u} \in \mathbb{R}^2$ (not necessarily of unit length), we write

$$f^{\mathbf{u}}_{\mathbf{v}} = \nabla_{\mathbf{u}} f(\mathbf{v}) := \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{v} + t\mathbf{u}) \Big|_{t=0}$$

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