ARTICLE IN PRESS
JID: CAG [m5G;July 31, 2017;13:37]
Computers & Graphics 000 (2017) 1–7
Contents lists available at ScienceDirect
Computers & Graphics
journal homepage: www.elsevier.com/locate/cag

Special Issue on CAD/Graphics 2017

Planar cubic G^1 and quintic G^2 Hermite interpolations via curvature variation minimization

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ARTICLE INFO

Article history:
Received 15 June 2017
Revised 5 July 2017
Accepted 6 July 2017
Available online xxx

Keywords:
Hermite interpolation
Geometric continuity
Fair curve
Curvature variation
Polynomial representation

ABSTRACT

Given two data points and the associated unit tangents, cubic G^1 Hermite interpolation is a simple and efficient scheme to construct fair curves by optimizing certain energy functionals. In order to obtain shape-preserving interpolation desired for applications, this paper presents cubic G^1 Hermite interpolation by minimizing curvature variation energy subject to a feasible region, with the advantage of handling arbitrary G^1 data. As a result, the G^1 interpolating curves can always maintain specified end tangent directions by restricting the two parameters provided by G^1 constraint to be positive; and the numerical solution is obtained by an iterative algorithm using the block coordinate descend method. This approach can be further extended to quintic G^2 Hermite interpolation for input G^2 data. A number of comparative experiments are conducted to verify the applicability and effectiveness of the proposed method.

1. Introduction

The construction of fair curves is an important issue in computer aided design (CAD) and related application fields [1–4]. Although it is difficult to characterize the "fairness" of curves in a quantitative way, a general approach to constructing fair curves is realized by minimizing an energy functional representing the fairness, subject to the prescribed geometric constraints. Because curvature is the universal shape measure for curves, strain energy (SE) (also called bending energy) and curvature variation energy (CVE) are two widely adopted metrics of fairness [1,5], which are respectively defined for a planar curve $\mathbf{b}(t)$, $t \in [0, 1]$ by

$$\mathcal{E}_{SE}(\mathbf{b}) = \int_0^1 [\kappa(t)]^2 \, dt \quad \text{and} \quad \mathcal{E}_{CVE}(\mathbf{b}) = \int_0^1 [\kappa'(t)]^2 \, dt \qquad (1)$$

where $\kappa(t) = \frac{\mathbf{b}'(t) \times \mathbf{b}''(t)}{\|\mathbf{b}'(t)\|^3}$ is the curvature of $\mathbf{b}(t)$. Throughout this paper, the scalar cross-product, $\mathbf{x} \times \mathbf{y} := x_1 y_2 - x_2 y_1$, is used to give a compact expression for some formulas.

Geometric Hermite interpolation first introduced by de Boor et al. [6] deals with the interpolation of geometric data such as positions, unit tangents, and curvatures. In contrast to the classical Hermite interpolation, this approach allows additional parameters that can be used to produce more pleasant shapes by direct assignments or optimization techniques; geometric continuity is preferred in various applications such as shape modeling, because it is independent of parameterization and is merely a relaxation of parameterization (but not a relaxation of smoothness). It was shown in [6] that a planar cubic curve can interpolate the input G^2 data by assigning the four parameters to suitable values, but the numerical solutions exist only when admissibility conditions on the data are met. Another kind of methods represents a curve using more than enough degrees of freedom to satisfy geometric constraints or specific desires on its shape, with the remaining degrees of freedom determined in an optimal way. For example, a cubic G^1 interpolating curve always has two degrees of freedom, whose values are more suitably determined by minimizing an energy functional (see e.g. [7–10]). Jaklič and Žagar [8,9] proposed two elegant schemes by minimizing the approximate strain energy and the approximate curvature variation energy respectively, and showed the asymptotic behavior of both schemes. However, it is important to note that for any polynomial curve other than a straight line, these approximate energies are not identical to their actual counterparts and may usually lead to unsatisfactory results. Recently, Lu [11] constructed a quintic curve model with four free parameters for G^2 interpolation.

Deng and Ma [12] proposed a biarc-based subdivision scheme for generating planar spirals without any condition on the input G^2 data; but the subdivision spirals are non-polynomial. Yang [13] studied G^1 Hermite interpolation using logarithmic arc splines, where a practical algorithm is developed for computing the solutions. Also, Wu and Yang [14] presented G^1 and G^2 Hermite interpolations by using some intrinsically defined planar curves whose

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http://dx.doi.org/10.1016/j.cag.2017.07.007
0097-8493/

Please cite this article as: L. Lu et al., Planar cubic G^1 and quintic G^2 Hermite interpolations via curvature variation minimization, Computers & Graphics (2017), http://dx.doi.org/10.1016/j.cag.2017.07.007

curvature radius functions are low-degree polynomials in terms of the tangent angle. The interpolating curves obtained by [14] admit explicit expressions for the offset curves and become spirals when the conditions on input data are satisfied; but they are not represented in polynomial form.

Circles and circular arcs, having a simple expression and constant curvature, are widely used in the CAD literature and also play a critical role in representing a myriad of parts used in manufacturing. Farin [5] proposed G^1 Hermite interpolation with circular precision, by using a rational cubic curve model. Later, Li et al. [15,16] constructed C-shaped G^2 Hermite interpolation with conic precision. However, the rational representation may suffer from several inherent drawbacks. Hu [17] proposed an explicit and effective method for G^1 approximation of conic sections using polynomial curves of arbitrary degree.

In this paper, we study planar cubic G^1 and quintic G^2 Hermite interpolations by minimizing curvature variation energy. According to Farin [1], curvature variation energy is a more suitable and acceptable metric for producing aesthetically pleasant shapes since the variation in curvature is regarded as more important than the magnitude of curvature. Unfortunately, for any polynomial curve, its curvature variation energy is always highly nonlinear; therefore, we propose a numerical approach for solutions, based on numerical integration.

For cubic G^1 Hermite interpolation, we formulate the problem as a constrained minimization problem subject to a user-defined feasible region. Compared to the previous method [9] based on approximate curvature variation minimization, the actual curvature variation energy is used here. By restricting the two parameters (α_0, α_1) provided by G^1 constraint within the feasible region, we are able to match arbitrary G^1 data without any condition, while meeting the shape-preserving requirement. Consequently, our method is application-independent; whereas the previous methods [5,7–9] do not always guarantee a positive solution of (α_0, α_1), which may restrict their applications in shape design.

For quintic G^2 Hermite interpolation, the problem is manipulated analogously to the cubic G^1 case, but with four parameters $(\alpha_0, \alpha_1, \beta_0, \beta_1)$. In [11], after expressing β_0 and β_1 as quadratic functions of (α_0, α_1), the approximate strain energy is further simplified to a quartic function of (α_0, α_1). Whereas curvature variation energy used here is expressed in four unknowns and has to be calculated through numerical integration. Thus a higher computational cost is required.

The remainder of this paper is organized as follows. We present cubic G^1 Hermite interpolation in Section 2 and quintic G^2 Hermite interpolation in Section 3. In Section 4 we show many examples in

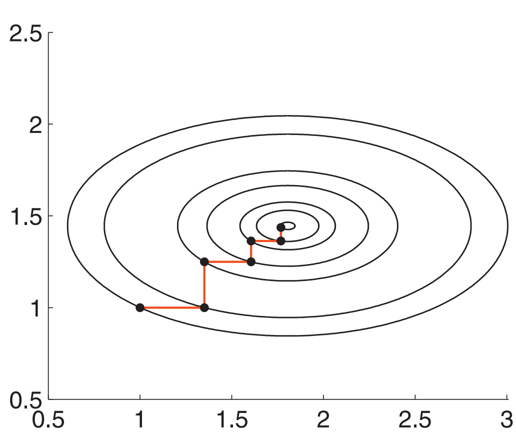

Fig. 1. An example of search path of (α_0, α_1).

Table 1
Curvature variation energy and computation time for the resulting curves in Fig. 2. (Note: the symbol "–" indicates that it is negligible in computation time.)

Fig.	Method [9]		This paper	
	$\mathcal{E}_{\rm CVE}$	Time	$\mathcal{E}_{\rm CVE}$	Time (s)
(a)	7.418	–	0.024	0.152
(b)	171.851	–	9.372	0.161
(c)	45.163	–	28.158	0.157
(d)	505.795	–	12.183	0.165
(e)	N/A	N/A	7.614	0.154
(f)	N/A	N/A	111.146	0.166

comparison with the previous methods. Finally, we conclude the paper in Section 5.

2. Cubic G^1 Hermite interpolation via curvature variation minimization

Given G^1 Hermite data $\{\mathbf{P}_0, \mathbf{T}_0, \mathbf{P}_1, \mathbf{T}_1\}$ represented by the positions and unit tangents at two points, a cubic G^1 interpolating curve is represented in Bézier–Bernstein form by

$$\mathbf{b}(t) = \sum_{i=0}^{3} \mathbf{b}_i B_i^3(t), \quad t \in [0, 1], \tag{2}$$

where $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$ are the Bernstein polynomials of degree n and the control points $\mathbf{b}_i \in \mathbb{R}^2$ are expressed as

$$\mathbf{b}_0 = \mathbf{P}_0, \quad \mathbf{b}_1 = \mathbf{P}_0 + \frac{\alpha_0}{3}\mathbf{T}_0, \quad \mathbf{b}_2 = \mathbf{P}_1 - \frac{\alpha_1}{3}\mathbf{T}_1, \quad \mathbf{b}_3 = \mathbf{P}_1, \tag{3}$$

with $\alpha_0, \alpha_1 > 0$ being real parameters.

Jaklič and Žagar [9] obtained the optimal value of (α_0, α_1) by minimizing $\int_0^1 [\mathbf{b}'(t) \times \mathbf{b}'''(t)]^2 \, \mathrm{d}t$, and the unique global minimum is reached at

$$\alpha_0 = -2\frac{\mathbf{T}_1 \times (\mathbf{P}_1 - \mathbf{P}_0)}{\mathbf{T}_0 \times \mathbf{T}_1}, \quad \alpha_1 = 2\frac{\mathbf{T}_0 \times (\mathbf{P}_1 - \mathbf{P}_0)}{\mathbf{T}_0 \times \mathbf{T}_1}. \tag{4}$$

However, α_0 and α_1 are guaranteed to be positive if and only if $\mathbf{T}_0$ and $\mathbf{T}_1$ are located at opposite sides of the straight line $\mathbf{P}_0\mathbf{P}_1$ and with a rotation angle less than π.

In this paper, we determine (α_0, α_1) by minimizing the curvature variation energy; that is, the problem is formulated as

$$\min_{(\alpha_0,\alpha_1)\in D} \mathcal{E}_{\rm CVE}(\mathbf{b}) := \int_0^1 [\kappa'(t)]^2 \, \mathrm{d}t. \tag{5}$$

In order to obtain positive solutions for shape-preserving interpolation demanded by various applications, it is important and practically useful to impose a feasible region on (α_0, α_1):

$$D = \left\{ (\alpha_0, \alpha_1) \in \mathbb{R}^2 \,|\, 0 < l_0 d \le \alpha_0 \le u_0 d, \;\; 0 < l_1 d \le \alpha_1 \le u_1 d \right\}, \tag{6}$$

where $d = \|\mathbf{P}_1 - \mathbf{P}_0\|$, l_i and u_i are user-specified lower and upper bounds. Noting that $\alpha_0 = \|\mathbf{b}'(0)\|$ and $\alpha_1 = \|\mathbf{b}'(1)\|$ are closely related to the magnitudes of end tangents, it is geometrically intuitive and meaningful to adjust the bounds of D and then to affect curve shape. In terms of the feasible region, the resulting curve is forced to maintain the specified tangent directions at the endpoints. Moreover, it is even beneficial to seek a solution that lies in a small region. Noting that α_0 and α_1 should not be too small or too large, $D = [0.2d, 5d]^2$ is suggested for the feasible region; in addition, it may be interactively adjusted if necessary.

The derivative of curvature yields

$$\kappa'(t) = \frac{\psi(t)}{\|\mathbf{b}'(t)\|^5}, \tag{7}$$

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