Computers & Graphics 54 (2016) 28–37

Contents lists available at ScienceDirect

Computers & Graphics

journal homepage: www.elsevier.com/locate/cag

Special Issue on CAD/Graphics 2015

Topological analysis for 3D real, symmetric second-order tensor fields using Deviatoric Eigenvalue Wheel

Xianyong Liu, Lizhuang Ma *

School of Computer Science and Technology, Shanghai Jiao Tong University, Shanghai, China

ARTICLE INFO

Article history:
Received 22 March 2015
Received in revised form
11 July 2015
Accepted 12 July 2015
Available online 21 July 2015

Keywords:
Tensor fields
Visualization
Topology
Interaction technique

ABSTRACT

Tensor fields are of particular interest not solely because they represent a wide variety of physical phenomena but also because crucial importance can be inferred from vector and scalar fields in terms of the gradient and Hessian, respectively. This work presents our key insights into the topological structure of 3D real, symmetric second-order tensor fields, of which the central goal is to show a novel computation model we call Deviatoric Eigenvalue Wheel. Based on the computation model, we show how the eigenanalysis for tensor fields can be simplified and put forward a new categorizer for topological lines. Both findings outperform existing approaches as our computation depends on the tensor entries only. We finally make a strict mathematical proof and draw a conclusion of $R = \mathbb{K}\ \cos(3\alpha)$, wherein R is the tensor invariant of determinant, $\mathbb{K}$ is a tensor constant, and α is called Nickalls angle. This conclusion allows users to better understand how the feature lines are formed and how the space is divided by topological structures. We test the effectiveness of our findings with real and analytic tensor fields as well as simulation vector fields.

1. Introduction

Visualizing 3D real, symmetric second-order tensor fields has found uses in various engineering contexts. The stress tensors for bone implantation in orthopaedics [1], the strain tensors for exhibiting isotropic responses in materials [2], the real, symmetric traceless tensors for simulating molecules in Nematic Liquid Crystals [3], and the diffusion tensors for monitoring the white matter structures in Magnetic Resonance Imaging [4,5] are a few examples.

Differing from conventional means such as Hyperstreamlines [6], Tensorlines [7], HyperLIC [8], HOTlines [9], and Superquadric glyphs [10], our work centers around the *topological analysis* of 3D real, symmetric second-order tensor fields (referred to here simply as “tensor fields”, unless otherwise stated). Despite recent progress towards topology-based visualization of tensor fields [11–14,5,15–17], the study aiming at bettering our understanding about topological structures and features of underlying tensor fields remains a challenge while being meaningful.

The reasons are threefold. First, in scientific data visualization the topology-based techniques, pursing the idea of showing less can be more, are proven to yield simplified, yet compact, depictions. Second, trained users can even reconstruct the data fields by looking at the topological structures in most cases. Third, the comprehension to scalar/vector fields can be improved through the topological analysis of tensor fields. For example, by applying a data transformation, vector and scalar fields are mapped into tensor fields with respect to the gradient and Hessian, respectively. As a result, many appealing representations for the datum, which are hardly speculated from the preimages directly, can be inferred from the images of tensor fields. As shown in Fig. 1 (right), where the simulated 3D dynamical Lorenz attractor system is visualized using our approach in the transformed tensor field, the degenerate curves are symmetric and behave like the pivots of two vortices, which the flow trajectories are around.

Early work on topology-based methods for visualizing tensor fields has laid an important background of this research project. Nevertheless, none of those techniques are flawless. It was showed that in 3D tensor fields double degeneracies are stable and form degenerate curves that could be detected by the “Extracting-and-Tracing” framework, however a fact that deviatoric (tensor) fields are topology-preserving was not sufficiently appreciated in [19,14,13]. Ref. [5] borrowed the result from [2] to regard degenerate lines of a tensor field as crease lines of the tensor invariant *mode*, unfortunately a strict mathematical proof to the result was not given by both. Moreover, the topological analysis about the essence of deviatoric fields was not explored either. Recent work of [17] went deeper and gave many interesting results, such as the traceless surface and the neutral surface, however its proposal of the eigenvalue manifold was not as

* Corresponding author.
E-mail address: ma-lz@cs.sjtu.edu.cn (L. Ma).

http://dx.doi.org/10.1016/j.cag.2015.07.009
0097-8493/

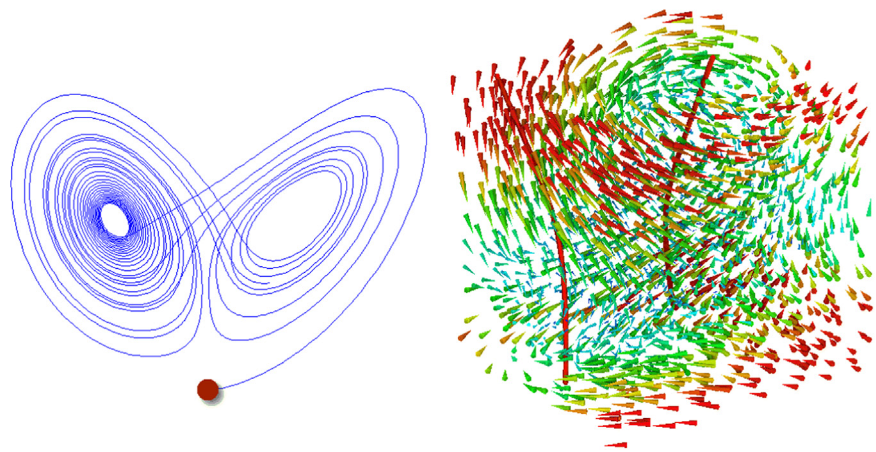

Fig. 1. Left: the Lorenz attractor (a.k.a. the Lorenz butterfly) of a simulated 3D dynamical system that exhibits chaotic flow [18] ($rho = 28; sigma = 10; beta = 8.0/3.0; t_i = [0, 25]$). Right: The topological lines of the tensor field transformed from the flow field (t_i=0, and the glyphs describe the vector field direction).

straightforward as the eigenvalue wheel in [20,5]. Additionally, the approach for detecting degenerate curves in [17] not only was time-consuming but also encountered undesirable zig-zags.

The central goal of this paper is to show a novel computation model we call *Deviatoric Eigenvalue Wheel* (DEW, demonstrated in Fig. 3), with which this paper makes the following contributions.

Eigenanalysis: Conventional eigendecomposition has been widely used in visualization, yet it is expensive and sensitive to numerical computations. The presented eigenanalysis is faster and more robust, as its computation relies on the tensor entries only (see Section 8.1).

Categorization: The categorization of feature lines accounts for reflecting how the fields are divided by topological structures. The concept of *eigen difference* was employed by [19,14,13], however interpolating colors failed to inform the real type of feature lines. Regarding that triple degeneracies in tensor fields are rare and unstable, [5] and [17] suggested to categorize topological lines by applying the extremal values of the *mode* and the *LP Factor*, respectively. In the paper, we put forward a new categorizer that depends on the sign of tensor determinant.

Tensor invariant: Tensor invariants are closely related to analyzing the topological features of tensor fields. Ref. [5] pointed out that the degenerate curves are actually the crease (ridge and valley) lines of the tensor invariant *mode*, where the computations took advantage of the result in [2] while lacking a strict mathematical proof. Such a pitfall is overcome in this paper.

2. Previous work

This section mentions the previous work highly relevant to this paper. For general tensor fields, we refer the interested readers to [21,6,22,15,16].

The salient topological features in 2D tensors fields are degenerate points [23], i.e., trisectors and wedges, whose *tensor indices* are half-integers due to the indeterminacy of eigendirections that is not seen in regular vector fields. The understanding of topology of 3D tensor fields remained vague until [24] provided some useful findings, e.g., the "double degeneracy" and "triple degeneracy". Unfortunately, [24] did not recognize that in 3D cases the topological formalisms are continuous lines instead of isolated points.

Rewardingly, using the theory of *dimensional analysis*, Zheng and Pang [19] proved that the triple degeneracy in 3D tensor fields is actually unstable and typically absent from practical datasets, conversely the double degeneracy is stable and constitutes one dimensional *degenerate lines* (a.k.a. *degenerate curves*, *feature lines*, or *topological lines*). The local behaviors nearby degenerate points were unclear until [14] attempted this issue and experimentally showed that transition points are exactly the points where the degenerate tensors switch types between trisectors and wedges. While figuring out analytic equations for the transition points is still an open problem in tensor field analysis.

Another local behavior is the tangent along degenerate curves which was fully discussed in [13] by using the Taylor's expansion of the well-known *tensor discriminant*, showing that the tangent around a degenerate point is just the direction from which the discriminant's Hessian vanishes, so do itself and its gradient. It was complained that the algorithms in [19,13] produced results that were sensitive to noise and essentially meaningless in the context of Diffusion Tensor Imaging (DTI) [4]. To overcome this difficulty, [5] changed our mindset by regarding the degenerate curves as the crease (ridge and valley) lines of the invariant *mode* defined as $\sqrt{2}\mu_3/\sqrt{\mu_2}^3$, wherein $\mu_1 = \sum_i \lambda_i/3$, $\mu_2 = \sum_i (\lambda_i - \mu_1)^2/3$, and $\mu_3 = \sum_i (\lambda_i - \mu_1)^3/3$ are the central tensor moments, and λ_i are the tensor eigenvalues.

3. Fundamentals of tensor topology

A 3D real, symmetric second-order *tensor field* is usually defined by a smooth tensor-valued function, associating each point in the field with a value $\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j (1 \le i, j \le 3)$, wherein T_{ij} are the *tensor entries* in a space spanned by the basis of dyads $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$. In tensor field visualization, it was acknowledged that a tensor filed is fully equivalent to three eigenvector fields ν_{1-3} satisfying $\mathbf{A}(\lambda)\nu_i = \mathbf{0}$ with three eigenvalues λ_{1-3} representing magnitudes. As is known, $\mathbf{A}(\lambda)$, i.e., $\lambda\mathbf{I} - \mathbf{T}$, is the *characteristic matrix*.

The roots of the polynomial $p(\lambda) = det(\mathbf{A}(\lambda))$ correspond to tensor eigenvalues that are ordered in a sequence of $\lambda_1 \ge \lambda_2 \ge \lambda_3$, in which the values $\lambda_i (1 \le i \le 3)$ are individually ranked as *major*, *medium*, and *minor*. So are the corresponding tensor eigenvectors $\nu_i (1 \le i \le 3)$. The coefficients of $p(\lambda)$ are defined as $P = \text{tr}(\mathbf{T})$, $J = (\text{tr}(\mathbf{T})^2 - \text{tr}(\mathbf{T}^2))/2$, and $R = det(\mathbf{T})$. They are all tensor invariants [26], among which P and R quantify the *trace* and *determinant*, respectively. Quantities of *minor* $Q = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$, *norm* $\|\mathbf{T}\| = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{1/2}$, and *tensor discriminant* $D_3 = (\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2$ are other important tensor invariants, which can be calculated via tensor entries [19].

The *topological structure* of tensor fields can be specified by tensor degeneracies and separatrices. Tensor degeneracies are formed by degenerate points that are the only places where hyperstreamlines can cross each other, and separatrices are the boundaries of hyperbolic sectors where trajectories sweep past degenerate points. As