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Radio cover time in hyper-graphs

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ABSTRACT

In recent years, protocols that are based on the properties of random walks on graphs have found many applications in communication and information networks, such as wireless networks, peer-to-peer networks, and the Web. For wireless networks, graphs are actually not the correct model of the communication; instead, hyper-graphs better capture the communication over a wireless shared channel. Motivated by this example, we study in this paper random walks on hyper-graphs. First, we formalize the random walk process on hyper-graphs and generalize key notions from random walks on graphs. We then give the novel definition of radio cover time, namely, the expected time of a random walk to be heard (as opposed to visited) by all nodes. We then provide some basic bounds on the radio cover, in particular, we show that while on graphs the radio cover time is $O(mn)$, in hyper-graphs it is $O(mnr)$, where n, m, and r are the number of nodes, the number of edges, and the rank of the hyper-graph, respectively. We conclude the paper with results on specific hyper-graphs that model wireless mesh networks in one and two dimensions and show that in both cases the radio cover time can be significantly faster than the standard cover time. In the two-dimension case, the radio cover time becomes sub-linear for an average degree larger than $\log^2 n$.

1. Introduction

Random walks are a natural and thoroughly studied approach to randomized graph exploration. A *simple random walk* is a stochastic process that starts at one node of a graph and at each step moves from the current node to an adjacent node chosen randomly and uniformly from the neighbors of the current node. It can also be seen as uniformly selecting an adjacent edge and stepping over it. Since this process presents locality, simplicity, low-overhead (i.e., memory space), and robustness to changes in the network (graph) structure, applications based on random-walk techniques are becoming more and more popular in the networking community. In recent years, different authors have proposed the use of random walk for a large variety of tasks and networks; to name but a few: querying in wireless sensor and ad hoc networks [4,22], searching in peer-to-peer networks [17], routing [8,23], network connectivity [11], building spanning trees [9], gossiping [18], membership service [7], network coding [14], and quorum systems [16].

Two of the main properties of interest for random walks are *hitting times* and the *cover time*. The hitting time between u and v, $h(u, v)$, is the expected time (measured by the number of steps or in our case by the number of messages) for a random walk starting at u to visit node v for the first time. The *cover time* C_G of a graph G is the expected time taken by a simple random walk to visit all the nodes in G. This property is relevant to a wide range of algorithmic applications, such as searching, building a spanning tree, and query processing. Methods of bounding the cover time of graphs have been thoroughly investigated [2,10,12,20,28], with the major result being that cover time is always at most polynomial for static graphs. More precisely, it has been shown by Aleliunas et al. [3] that C_G is always at most 2 mn, where m is the number of edges in

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the graph and n is the number of nodes. There are many graphs for which this bound if tight, i.e., the cover time is $\Omega(mn)$.

While simple graphs are a good model for point-to-point communication networks, they do not capture well shared channel networks like wireless networks and LANs. In wireless networks the channel is shared by many nodes; this, on the one hand, leads to contention, but on the other hand, can be very useful for dissemination of information. When a node transmits on the shared channel, all other nodes that share the channel can receive or "hear" the message. This situation, as noted in the past for other wireless network applications [19], should be modeled as a (directed) hyper-graph and not as a graph. Hyper-graphs are a generalization of graphs where edges are sets (or ordered sets) of nodes of arbitrary size. A graph is a hyper-graph with the size of edges equal to two for all edges. For example, in wireless networks, there is a (directed) hyper-edge from each transmitter to a set of receivers that can decode its message.

Another set of examples emerge from social networks, which are also traditionally modeled as graphs. Taking a closer look at "friendship" relations in such networks reveals that they are actually, in many cases, hyper-edges and not edges. For example, tagged pictures in Facebook, joint papers in DBLP, and movies in IMDB.

Motivated by the hyper-graph model for wireless networks, in this paper we study random walks on hyper-graphs. We extend the hitting time and cover time definitions to the case of hyper-graphs, and in particular wireless networks. We define the *radio hitting time* from u to v as the expected number of steps for a random walk (to be defined formally later) starting at u to be heard by v for the first time. A node can hear a message when the message is transferred on an edge to which the node belongs. Clearly, the radio hitting time is less than the hitting time, so it will give a tighter bound on the time to disseminate information between nodes using random walks on hyper-graphs. But, while hitting times are well studied on graphs, it is not clear, at first sight, how to compute radio hitting times. In a similar manner, we define the *radio cover time* as the expected number of steps for a random walk to be heard by all the nodes in the graph. Again, this will give a better bound on the time to spread information among all the nodes, for example, in a random-walk-based search or query.

To our surprise, we found almost no previous work on random walks on hyper-graphs, in particular not any theoretical work. Since our conference version [6], Cooper et. al. [13] also addressed this topic and provided bounds for the cover times of random hyper-graphs. One exception is the experimental study on simple random walks on hyper-graphs performed by Zhou et al. [27]. In that work, the authors suggest using a simple random walk on hyper-graphs to analyze complex relationships among objects by learning and clustering. Our work is general enough to make a contribution in this direction as well.

1.1. Overview of our results

The paper's contribution covers two main themes. In Section 3 we elaborate on definitions from [27] about simple random walks on hyper-graphs. We extend known parameters and properties of random walks on graphs to the case of hyper-graphs. We explore the relation between a random walk on the set of vertices of the hyper-graph to a random walk on the set of edges of the hyper-graph and between random walks on hyper-graphs and random walks on special bipartite graphs. We study the undirected and directed cases and base all our definitions on the basic object that describes a hyper-graph, *the incidence matrix*. Moreover, we formally define the novel notion of radio hitting time and radio cover time, as mentioned above; This formalism will be an essential tool in pursuing future research on random walks on hyper-graphs.

The second theme of the paper is to provide algorithms and to prove bounds for the main properties of interest for random walks. In Section 4 we study radio hitting times and in Theorem 1 we provide an algorithm to compute the radio hitting time on hyper-graphs. Section 5 studies radio cover times and presents general bounds. Theorem 2 addresses the maximum *speedup* that could be achieved in radio cover time. We then generalize famous bounds on the cover times of graphs to radio cover times on hyper-graphs: in Theorem 3 we extend Matthews' bound [20] to radio hitting time and in Theorem 4 we extend the fundamental bound on the cover time of Aleliunas et al. [3]. We extend the bound of $mO(mn)$ on the cover time of graphs by an $O(mnr)$ bound on the radio cover time of hyper-graphs, where n is the number of nodes, m is the number of edges, and r is the size of the maximum edge. Note that while for graphs, m is at most n^2, for hyper-graphs m could be exponential; we show that even in this case the bound could be tight. In Section 6.1 we study hyper-graphs that model wireless radio networks. Theorems 7 and 6 bound the expected time for all nodes in the network to "hear" the message in 1-dimensional and 2-dimensional grids, respectively. These results capture some of the nice properties of radio cover time; we show that as the sizes of hyper-edges increase, the radio cover decreases. But while cover time cannot go below $n \log n$, radio cover time can be much smaller, and become sub-linear when edges are large enough. Conclusions are then presented in Section 7.

2. Models and preliminaries

We now present formal definitions of the involved objects; in some cases we follow definitions taken from PlanetMath. A (finite, undirected) *graph* G is an ordered pair of disjoint finite sets (V, E) such that E is a subset of the set $V \times V$ of unordered pairs of V. The set V is the set of *vertices* (sometimes called nodes) and E is the set of *edges*. If G is a graph, then $V = V(G)$ is the vertex set of G, and $E = E(G)$ is the edge set. If v is a vertex of G, we sometimes write $v \in G$ instead of $v \in V(G)$. For a set x let $|x|$ denote the cardinality of the set. We follow with a formal definition for an undirected hyper-graph.

Definition 1 (*Hyper-graph*). A hyper-graph $\mathcal{H}$ is an ordered pair $(V, \mathcal{E})$, where V is a set of vertices, and $\mathcal{E} \subseteq 2^V$ is a set of hyper-edges between the vertices, i.e.,