Int. J. Electron. Commun. (AEÜ) 84 (2018) 199–205

Contents lists available at ScienceDirect

Int. J. Electron. Commun. (AEÜ)

journal homepage: www.elsevier.com/locate/aeue

Regular paper

Dynamics, implementation and stability of a chaotic system with coexistence of hyperbolic and non-hyperbolic equilibria

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ARTICLE INFO

Keywords:
Chaotic system
Non-hyperbolic equilibrium
Circuital implementation
Stabilization

ABSTRACT

Hyperbolic-type equilibrium requires that all the real parts of the corresponding eigenvalues are nonzero. In this paper, a three-dimensional autonomous chaotic system is introduced, and interestingly we find that one non-hyperbolic equilibrium point and two hyperbolic equilibrium points coexist in this system, which, according to the information we know, has not been previously reported. We first reveal the basic dynamics of the system through analyzing phase portrait, frequency spectrum, Poincaré map, bifurcation diagram and Lyapunov exponent. Then, based on the idea of the improved modular technology, we build an analog circuit to realize the chaotic system, which further verifies the theoretical results. Finally, we design a simple feedback controller on account of Lyapunov asymptotic stability theory, to globally suppress the system to its equilibrium points.

1. Introduction

The equilibrium point of dynamical system is defined as the constant solution of its differential equation. The study on the equilibrium's property of dynamical system has well served to explore the type of system [1–3], the shape of attractor [4–7], the amplitude variation of signal [8,9], and the practical engineering application [10–14]. If all the real parts of the corresponding eigenvalues are nonzero, the equilibrium is called to be hyperbolic [15]. For three-dimensional autonomous system, the hyperbolic equilibrium can be the type of stable or unstable saddle, node, saddle-focus or node-focus. Since the famous Lorenz model was found in 1963 [16], a number of Lorenz-type chaotic systems have been reported with two saddle-foci and one saddle [17–23]. They are hyperbolic-type system with nonzero real parts of characteristic roots, and the Šil'nikov homoclinic/heteroclinic theory is one commonly accepted criterion for proving the existence of chaos [24,25]. What's more, these systems can generate two-scroll attractors alternatively swirling around the unstable saddle-focuses, which also imply the fact that the distribution of scrolls is determined by the number of saddle-focuses. The subsequent investigation along this fact naturally extended to construct (multi-directional) multi-scroll attractors through increase in the number of hyperbolic-type equilibrium points with index-2, which can be realized by substituting the original nonlinear term with cubic function [26], polynomial function [27], multi-segment quadratic function [28], saw-tooth function [29], hysteresis function [30] and stair function [31].

At the same time, other few chaotic systems with non-hyperbolic equilibria, defined as holding eigenvalues with a real part equal to zero [32], are formulated and studied. For example, Li and Ou reported a chaotic system originated from Lorenz system and obtained the stability character of its non-hyperbolic equilibria by using the center manifold theorem [33]. Liu and Yang also investigated the condition of asymptotically stable of the non-hyperbolic equilibrium for a Lorenz-like chaotic system [34]. Wei and Yang constructed a new chaotic system coexistence with saddle-foci, non-hyperbolic and stable equilibria by varying one of system parameters [35]. As was found by Sprott [36,37], the saddle-focus equilibria doesn't exist in the non-hyperbolic type of chaotic system. Therefore, it's difficult for the Šil'nikov theorem to detect the chaos in these abnormal chaotic systems of non-hyperbolic type.

Motivated by the above discussion, this paper reports our recent work of a three-dimensional autonomous chaotic system which, particularly interesting, holds three uncommon equilibrium points: a zero equilibrium point of non-hyperbolic type and two symmetrical

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https://doi.org/10.1016/j.aeue.2017.12.001
Received 14 June 2017; Accepted 1 December 2017
1434-8411/

equilibrium points of hyperbolic type. The finding is striking since it reveals that hyperbolic type and non-hyperbolic type equilibrium points can coexist in the same chaotic system, which, according to the information we know, has not been previously reported. And the finding will also constitute a stimulus to explore more undiscovered dynamics feature of chaotic system. What's more, the proposed system will hold more complex topological form than stable manifold, unstable manifold and central manifold, compared to non-hyperbolic system, consequently it will enhance the potential application in chaotic cryptography and secure communication. To understand the complex behavior of the system, basic dynamical properties, such as phase portrait, frequency spectrum, Poincaré map, bifurcation diagram and Lyapunov exponent are studied. And to further verify the theoretical results, we build an analog circuit to realize the chaotic system based on the idea of improved modular technology. Finally, to globally suppress the system to its equilibrium points, we design a feedback control scheme based on the theory of Lyapunov asymptotic stability. This scheme is simple with only one linear controller yet impactful to suppress the system to its different equilibrium points. Consequently, the scheme is practicable in actual implementation, which is further illustrated by numerical simulations.

2. Model and dynamics of the reported system

2.1. The system model

Our reported dynamical system is depicted in the following form

$$\begin{cases}\dot{x}_1 = -x_1 + x_2x_3^2\\ \dot{x}_2 = ax_2 - x_1x_3\\ \dot{x}_3 = -x_3^3 + x_1x_2\end{cases} \quad (1)$$

In system (1), x_1, x_2, x_3 are the state variables, a is the system parameter with positive value. It's known that system (1) is invariant under the transformation $(x_1, x_2, x_3) \mapsto (-x_1, -x_2, x_3)$, thus we conclude that system (1) has reflection symmetry about x_3-axis, and the nontrivial trajectories of x_1 and x_2 of system (1) hold a twin direction.

When the only parameter a is equal to 2 and initial selection is set as $(x_1(0), x_2(0), x_3(0)) = (0.2, 0.1, 0.1)$, the three Lyapunov exponents of system (1) are calculated as $(0.210312, 0.004775, -6.435465)$ by the orthogonal method with the simulation time $T = 1000$. As we can deduce that system (1) is chaotic since it holds one positive Lyapunov exponent. And the fractional Kaplan-Yorke dimension is subsequently derived to be 2.0319, which also confirms the chaotic behavior. The corresponding phase diagrams further verify the chaotic property for the reported system, as depicted in Fig. 1.

In the frequency domain, we depict an apparently continuous broadband spectrum $20\log|x_2|$ of system (1) in Fig. 2(a). While in the time domain, we visualize the Poincaré map on x_1–x_3 plane with $x_2 = 0$ in Fig. 2(b). It is clear that the Poincaré map is composed of virtually symmetrical branches and several nearly symmetrical twigs. What's more, the attractor structure is displayed in the Poincaré map.

2.2. Analysis of equilibrium points

By solving the equation set $-x_1 + x_2x_3^2 = 0$, $ax_2 - x_1x_3 = 0$, $-x_3^3 + x_1x_2 = 0$, we find three equilibrium points of system (1) as $E_0(0, 0, 0)$, $E_1(a^{5/6}, a^{1/6}, a^{1/3})$, $E_2(-a^{5/6}, -a^{1/6}, a^{1/3})$.

When selecting $a = 2$, we obtain the equilibrium points and the corresponding eigenvalues, as follows:

$E_0(0, 0, 0)$: $\lambda_1 = -1$, $\lambda_2 = 0$, $\lambda_3 = 2$.
$E_1(1.7818, 1.1225, 1.2599)$: $\lambda_1 = 0.7826 + 1.7214i$, $\lambda_2 = 0.7826 - 1.7214i$, $\lambda_3 = -5.3272$.
$E_2(-1.7818, -1.1225, 1.2599)$: $\lambda_1 = 0.7826 + 1.7214i$, $\lambda_2 = 0.7826 - 1.7214i$, $\lambda_3 = -5.3272$.

Thus, equilibrium point E_0 is non-hyperbolic since the characteristic value λ_2 equals to zero. However, equilibrium points E_1 and E_2 are unstable saddle-focus of index 2. And since all the real parts of the corresponding eigenvalues are nonzero, the two nonzero equilibrium points are hyperbolic. This finding reveals the unusual fact that hyperbolic type and non-hyperbolic type equilibrium points can simultaneously exist in one chaotic system.

One might get the impression that the solutions of system (1) are actually bounded, and it has a globally attracting chaotic attractor. However, if one chooses $x_1(0) = x_3(0) = 0$ and $x_2(0) = c \neq 0$, then the exact solution of (1) is $x_1(t) = x_3(t) = 0$ and $x_2(t) = ce^{at}$. Hence the x_2 axis is actually an unstable manifold for (1) when $a > 0$. As a result, if initial conditions are close to this axis, solution bound will increase. Hence, for the equilibrium point E_0, if one chooses a $g > 0$, then there always exists initial conditions for which the inequality to determine g is violated. In other words, the stability result for E_0 is only valid for a set of initial conditions, hence is local in nature.

2.3. Dynamics switch by parameter variation

To reveal the process of dynamics switch for this system, we consider the parameter range $1 \le a \le 2.6$, and the corresponding bifurcation diagram and spectrum of Lyapunov exponents by numerical calculation are depicted in Fig. 3. Preliminary analysis shows that the dynamical behaviors of system (1) switch among chaotic orbit and periodic orbit connected by inverse period-doubling bifurcation, with the increase of parameter a. Concretely, there exist five obvious periodic windows embedded in the chaotic region, with parameter a belonging to $[1.195, 1.24]$, $[1.547, 1.573]$, $[1.686, 1.824]$, $[2.152, 2.372]$ and $[2.551, 2.6]$, respectively. As illustrated examples, we depict the periodic motion when a equal to 1.24 and 2.16 respective, seen in Fig. 4.

3. Circuit implementation of the reported system

From the point of practical applications, the hardware realization of chaotic models is an important topic, especially realized by using commercially common electronic components [38–40]. Therefore, we will design an electronic circuit to realize the reported chaotic system in this section, by using the dimensionless state equations and the improved module-based technique [41].

First, to ensure the dynamic range of state variables determined by saturation value of active devices, and to guarantee the electronic circuit working effectively and capture the wave easily, the variable-scale reduction and time-scale transformation should be taken into account. Thus, when letting the proportional compression factors be $(10, 2, 1)$ for variables (x_1, x_2, x_3), and letting the time-scale transformation factor be 100, we derive the resulted state equation of system (1) with $a = 2$, as below

$$\begin{cases}\dot{x}_1 = -100x_1 - 20(-x_2)x_3^2\\ \dot{x}_2 = -200(-x_2) - 500x_1x_3\\ \dot{x}_3 = -100x_3^3 - 2000x_1(-x_2)\end{cases} \quad (2)$$

Secondly, the improved module-based circuit diagram from state equation (2) can be derived by differential to integral conversion, as depicted in Fig. 5. In this design, we choose the operational amplifier LF353 chip and analog multiplier AD633 chip. It is worth mentioning that the AD633 may hold nonideal memory effect in actual implementation [42]. In the course of our implementation, the working condition is considered to be ideal for avoiding the ranges in which nonideal behavior is problematic. Thus, we obtain the circuit state equation from Fig. 5, as follows