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# Fractional electronic circuit simulation of a nonlinear macroeconomic model

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ABSTRACT

The Fractional Electronic Circuit Simulation (FrECS) using the Cadence OrCAD™ package is explored for evaluating the dynamics of a nonlinear macroeconomic system. The model of the dynamic system is translated into two forms, namely Numerical Simulation (NS) and FrECS, both for integer and non-integer orders. The results show not only a good agreement between the two simulation approaches, but also demonstrate the feasibility of using FrECS to investigate fractional dynamics. The FrECS is, therefore, a reliable strategy when having in mind a physical implementation using circuits.

## 1. Introduction

An analog computer uses continuous variables to represent those of a given physical phenomena. During the 40s analog computers were implemented using electric and electronic elements, and were capable of simulating systems such as those described by electrical, mechanical, or hydraulic models using analogies between the distinct domains [1,2]. In 1949, A.W. Phillips reflects on how a hydraulic analog computer can work by modeling an economic system through the so-called Phillips machine [3]. In this machine, the flow of money around an economy maps to the flow of liquid around the machine. At various points during the flow, proportions of liquid could be routed off into storage vessels representing, say, national savings. Government borrowing is represented by removing money from a storage vessel. The machine developed by Phillips provided the idea of a realization of a system of differential equations. It represented variable quantities as levels of fluid and produced an illustrative model of Keynesian economics.

The emergence of the digital computing made analog computers obsolete by the 50s and 60s with exception of some specific applications. During the 70s emerged the concept of electronic circuit emulation by means of digital computers [4]. Electronic circuit simulation replicates the behavior of a real-world electronic circuit. Simulation software allows for the circuit analysis before any hardware implementation and is a reliable tool both for analog and digital systems [5]. Since electronic circuit emulation is a well established technique, we can think of its application to simulate a system in any domain. This strategy is somehow the inverse of the initial one using analog computers. In fact, we adopt digital computers to simulate an analog system and we use the analogies between two domains in the inverse way. This means that we can adopt electronic circuit simulation to handle, for example, a mechanical system, using its analogy to the electrical domain. This paper investigates this strategy using a well-known electronic circuit simulator with a non-linear economy model, both for integer and fractional orders.

Fractional calculus (FC), the theory of derivatives and integrals of non-integer order, follows a long history in mathematical analysis and has presently a wide applications in dynamical systems [6–8]. FC can represent complex dynamics systems using merely a few coefficients, since the derivative orders provide additional degrees of freedom to adjust models to the phenomena [9]. Derivatives and integrals of non-integer order have geometric [10], probabilistic [11,12] and discrete interpretations [13,14]. The properties of FC allow the description of complex dynamical phenomena, namely systems and media characterized by long-term memory and power law behavior, present in economic and financial systems [15]. We can find various mathematical definitions involving fractional derivatives such as those formulated by Riemann, Liouville, Caputo, Laurent, Grünwald, Letnikov, Marchaud and others [16,17]. Nevertheless, these distinct perspectives do not limit the application of FC in real-world phenomena and, in fact, allow researchers to choose the definition that fits more adequately to the phenomenon under study.

The analysis of economic and financial systems requires accurate models and several researchers proposed the application of FC to evaluate and understand their dynamic behavior [18–24]. Chen [25] considered a generalization of the system proposed by Ma and Chen [26] for fractional orders. Li and Peng [27] showed the strong presence of chaos in Chen's system with fractional order. David et al. [28]

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proposed a model of fractional order involving the public sector deficit. The fourth-order system of fractional differential equations with coupled variables was analized by means of NS.

Electronic circuits can be adopted to explore nonlinear dynamics, time series, bifurcation diagrams, chaotic behavior, engineering applications involving communications, signal processing and control [29–33]. In fact, electronic circuit simulation has been used as an alternative to the electronic physical implementation and/or numerical simulations for evaluating nonlinear dynamical systems. However, only a few works deal with the electronic circuit simulation of fractional dynamical systems [34–40] and a considerable room of research still remains open.

Tacha et al. [41] investigated a third-order financial system of integer order using electronic circuit simulation. The results emulate adequately the behavior of the dynamical system and compare well with numerical integration methods. Hajipour and Tavakoli [42] emulated a fractional-order financial system using FrECS considering a specific commensurate order and achieved results close to those obtained by means of NS.

In this work, the FrECS technique using the Cadence OrCAD™ package is proposed to evaluate the dynamic behavior of one fractional order macroeconomic system reported recently [28]. The results achieved by NS and FrECS are compared and discussed.

This paper is organized as follows. Section 2 develops the model, the circuit design and its implementation. Section 3 discuss the results obtained by means of NS and FrECS. Finally, Section 4 outlines the main concluding comments.

## 2. Model and circuits implementation of the fractional-order nonlinear macroeconomic system

FC is a generalization of the integration and differentiation toward of a non-integer order operator ${}_aD_t^{\alpha}$, where $a$ and $t \in R$ denote the lower and upper bounds of integration, $\alpha$ represents the order of operation and $Re(\alpha)$ is the real part of $\alpha$, such that:

$$ {}_aD_t^{\alpha} f(t) = \begin{cases} \frac{d^{\alpha} f(t)}{ft^{\alpha}} & Re(\alpha) > 0 \\ 1 & Re(\alpha) = 0 \\ \int_a^t f(t)(d\tau)^{-\alpha} & Re(\alpha) < 0. \end{cases} \tag{1} $$

The Riemann-Liouville (RL) approach [16,17] follows the definition of integration of arbitrary order

$$ \frac{d^{\alpha}}{dx^{\alpha}}[{}_cD_x^{-\rho} f(x)] = \frac{d^{\alpha}}{dx^{\alpha}}\left[\frac{1}{T(\rho)}\int_c^x (x-t)^{\rho-1} f(t)dt\right]. \tag{2} $$

In this study the macroeconomic system (3) modeled by David et al. [28] that uses the RL formulation is considered.

Electronic circuits are designed using the Cadence OrCAD™ package for representing integer and non-integer orders of derivatives adopted in the model. System (3) consists of a set of nonlinear differential equations with arbitrary order derivatives and four coupled variables $X$ (interest rate), $Y$ (investment demand), $Z$ (price index) and $W$ (public sector deficit)

$$ \begin{aligned} \frac{d^{q_1}X}{dt^{q_1}} &= Z + (Y-a)X + W, \\ \frac{d^{q_2}Y}{dt^{q_2}} &= 1-bY-X^2-W, \\ \frac{d^{q_3}Z}{dt^{q_3}} &= -X-cZ-W, \\ \frac{d^{q_4}W}{dt^{q_4}} &= (W-d)X-Y, \end{aligned} \tag{3} $$

**Table 1**
The three cases under study and the values of $\alpha$.

| Case | Order | Values of $\alpha$ |
|---|---|---|
| I | $q_1 = q_2 = q_3 = q_4 = \alpha$ | $\alpha = 1$ |
| II | $q_2 = q_3 = q_4 = 1$ and $q_1 = \alpha$ | $\alpha = 0.7$, 0.8 and 0.9 |
| III | $q_1 = q_2 = q_4 = 1$ and $q_3 = \alpha$ | $\alpha = 0.8$ and 0.9 |

where the operator $D_t^{q_i} = \frac{d^{q_i}}{dt^{q_i}}$, $i = 1,\ldots,4$, denotes the fractional derivative of order $q_i \in R$. The parameters $a, b, c, d \in R^+$ represent: $a$ saving amount; $b$ cost per investment; $c$ elasticity of demand of commercial markets and $d$ cost of public debt.

We start by simulating system (3) by means of NS and FrECS. For comparing the NS results with those presented in [28], the orders of the fractional derivatives are those shown in Table 1. We apply the RL formulation, Runge Kutta solver and the Adams scheme in order to proceed with the NS of the system (3).

In a macroeconomic model, the decision to vary the Investment ($Y$) and to increase or decrease the public sector deficit ($W$) can be taken by the managers of the economic policy. Therefore, we can consider inputs involving the variables $Y$ and $W$ in system (3). Taking this effect into account, that is, by including both inputs, $u_Y(t)$ and $u_W(t)$, we obtain the following generalization

$$ \begin{aligned} \frac{d^{q_1}X}{dt^{q_1}} &= Z + (Y-a)X + W, \\ \frac{d^{q_2}Y}{dt^{q_2}} &= 1-bY-X^2-W + u_y(t), \\ \frac{d^{q_3}Z}{dt^{q_3}} &= -X-cZ-W, \\ \frac{d^{q_4}W}{dt^{q_4}} &= (W-d)X-Y + u_w(t), \end{aligned} \tag{4} $$

that is also analyzed and simulated using the FrECS.

For solving the equations by means of electronic circuits and to obtain a direct implementation using the inverting integrator configuration with operational amplifiers, one can rewrite system (4) in the form

$$ \begin{aligned} X &= -D_t^{-q_1}[-Z-XY + aX-W], \\ Y &= -D_t^{-q_2}[-1 + bY + X^2 + W + u_y(t)], \\ Z &= -D_t^{-q_3}[X + cZ + W], \\ W &= -D_t^{-q_4}[-WX + dX + Y + u_w(t)]. \end{aligned} \tag{5} $$

The FrECS of the system (5) is illustrated in Fig. 1. Each equation is solved by means of electronic inverting integrators that are implemented using the TL084 operational amplifiers U1A, U1B, U1C, and U1D, combined with feedback bipoles $Z_X, Z_Y, Z_Z$ and $Z_W$, and several resistors. To implement the integer or fractional order integration, the feedback bipoles $Z_X, Z_Y, Z_Z$ and $Z_W$ are replaced by capacitors or $RC$ lattices, respectively. The schematic diagram for the lattices is illustrated in Fig. 2.

The initial conditions of the variables $X, Y, Z$ and $W$ are implemented with the voltage sources $V_{j0}$ and switches $SW_j, j = X, Y, Z, W$. The product and the square operations are implemented using the AD633 integrated circuit UM1, UM2 and UM3. The inversion operations are implemented with the operational amplifiers U2A and U2B. The sinusoidal and step inputs, when considered, are implemented using voltage generators $v_Y(t)$ and $v_W(t)$.

The aforementioned lattices are synthesized based on the structure proposed in [41,42]. From these lattices one can obtain the transfer