Journal of Process Control 41 (2016) 47–55

Contents lists available at ScienceDirect

Journal of Process Control

journal homepage: www.elsevier.com/locate/jprocont

Robust stabilization of positive linear systems via sliding positive control

Horacio Leyva[a], Francisco A. Carrillo[a], G. Quiroz[b,*], R. Femat[c]

[a] Universidad de Sonora, Rosales and Transversal, Hermosillo, Sonora, Mexico
[b] Universidad Autónoma de Nuevo León, UANL, FIME, Av. Universidad S/N, Ciudad Universitaria, San Nicolás de los Garza, Nuevo León C.P. 66451, Mexico
[c] Division de Matemáticas Aplicadas, IPICYT, Camino a la Presa San José 2055, Col. Lomas 4a sección, C.P. 78216 San Luis Potosí (S.L.P.), Mexico

ARTICLE INFO

Article history:
Received 7 June 2015
Received in revised form 22 October 2015
Accepted 2 March 2016
Available online 21 March 2016

Keywords:
Positive systems
Compartmental systems
Sliding mode control

ABSTRACT

In this paper a robust control is proposed for a family of positive and compartmental systems. Sufficient conditions are provided for the stabilization of this kind of systems by using sliding mode theory. The construction of a stabilizing hyperplane with a sliding dynamics is detailed and the feasibility of the method is discussed. The method is illustrated with three examples. The first one is a two-dimensional system which is used only to show the details about the computation, the construction of the stabilizing hyperplane and the robustness of the control. Complementary, the last two are actual interesting cases of biomedical systems and they show potential applications about the stabilization and closed-loop performance. It should be noted that these biomedical systems arise as a current class of dynamical systems with interesting challenges for the process control.

1. Introduction

Many economical, physical, and biological systems involve quantities that are represented by positive variables, that is, concentration of substances, the liquid level in tanks or biomass. These examples belong to the class of positive systems, where the state variables and initial conditions are non negative, this kind of systems admits positive controls, by example in reactors and bio processes the control action is related to flows whose value are strictly positive, see [1,2]. In addition, this class of systems are often structured in compartments, as chemical reactors, physiological models, mixing tanks, etc. Due to this practical issue, there is a motivation on analysis of these control systems. A family of linear affine positive systems is considered in this paper, these systems satisfy the hypothesis of the theory of stability, such as the Frobenius–Perron theorem for Metzler matrices and Gerschgorin theorem applied to compartmental matrices. The main interest is to propose the existence of a sliding dynamic on a hyperplane segment of a general linear system with positive control. That is, let be the next a linear system with positive control

$$\dot{x} = Ax + bu, \tag{1}$$

where $x,\ b \in \mathbb{R}^n_+$ and $u \in [r_1, r_2] \subset \mathbb{R}_+$. The hyperplane segment of dimension $n-1$ is contained in $\mathbb{R}^n_+$ and, with an appropriated control, the stabilization rate of the system (1) can be improved.

2. Preliminaries

2.1. Positive control systems

A linear system in continuous time

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{2}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{r\times n}$ and $D \in R^{r\times m}$ is considered to be a positive linear system if for every nonnegative initial state and for every nonnegative input, the state of the system and the output remain nonnegative [3,4]. In this proposal, the existence of the sliding dynamics of the positive control system arises from some interesting properties of the homogeneous linear system:

$$\dot{x} = Ax, \tag{3}$$

where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. The next definitions and theorems are the main assumptions of this work.

* Corresponding author. Tel.: +52 8183294020x5773.
E-mail addresses: hleyva@mat.uson.mx (H. Leyva), carrillo@mat.uson.mx (F.A. Carrillo), griselda.quirozcm@uanl.edu.mx (G. Quiroz), rfemat@ipicyt.edu.mx (R. Femat).

http://dx.doi.org/10.1016/j.jprocont.2016.03.001
0959-1524/

Definition 1. The system (3) is called positive if $x(t, x_0) \geq 0$ for all $t \geq 0$ and $x_0 \geq 0$.

In other words, if $\mathbb{R}^n_+$ is an invariant set for the system (3), then the system is positive.

Definition 2. A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called Metzler matrix, if $a_{ij} \geq 0$ for $i \neq j$.

Theorem 1. *The system (3) is positive if and only if A is Metzler.*

Remark 1. A linear control system given by Eq. (2) is positive if and only if the matrix A is a Metzler matrix, and B, C, and D are nonnegative matrices.

Definition 3. A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called Hurwitz matrix, if all of its eigenvalues have negative real part.

Theorem 2. **Frobenius–Perron for Metzler matrices.**
Let A be a Metzler matrix. Then, there are a real number μ_0 and a vector $\omega_0 \geq 0$ such that,

(i) $A\omega_0 = \mu_0\omega_0$.
(ii) *If $\mu \neq \mu_0$ is an eigenvalue of A, then* $\mathrm{Re}(\mu) < \mu_0$.

Theorem 3. *Let $A \in \mathbb{R}^{n\times n}$ be a Metzler matrix. There is a positive matrix $-A^{-1}$ if and only if, A is a Hurwitz matrix, where A^{-1} is the inverse matrix of A.*

Theorems 1–3 are well known, and they define the conditions of positivity. Their demonstrations can be seen in [5,6].

2.2. A strategy to improve the stabilization rate with sliding mode theory

Consider the system (1), with $A \in \mathbb{R}^{n\times n}$ Metzler and Hurwitz, $b \in \mathbb{R}^n_+$, and $r_2 > r_1 \geq 0$. The positive fixed point $\bar{x} = -A^{-1}b\bar{u}$ (with a constant $\bar{u} \in [r_1, r_2]$) is a global asymptotically stable point. This leads to the question: is it possible to improve the stabilization rate, that is, the rapid convergence to $\bar{x}$ considering $u \in [r_1, r_2]$ instead of $u = \bar{u}$? Before answering this question, observe that the system (1) is not controllable, according to the controllability theorem of Brammer (see [7]), if A has at least a real eigenvalue, then the system (1) is not completely controllable with positive control.

Now, consider the system (1), where the matrix A is Metzler, the control $u \in [r_1, r_2]$, $r_2 > r_1 \geq 0$, and $b \in \mathbb{R}^n_+$. The next positive fixed points are considered:

$$\bar{x}_1 = -A^{-1}br_1 \quad \text{and} \quad \bar{x}_2 = -A^{-1}br_2$$

such that $\|\bar{x}_1\| < \|\bar{x}_2\|$. These equilibrium points are collinear with the origin.

The Hurwitz matrix A implies that each fixed point $\bar{x}_i$ is a globally attractor of the feedback system $\dot{x} = Ax + br_i$, $i = 1, 2$. To describe the sliding, the hyperplane in R^n_+ is firstly characterized. This is represented by the equation

$$Lx = k \tag{4}$$

where L is a row-vector in R^n and k is a scalar such that $k > 0$; both are parameters to be determined. The sliding condition is expressed by the inequalities:

$$\lim_{(Lx-k)\to 0^+} L(Ax + br_1) < 0 \quad \text{for} \quad x \in \mathbb{R}^n_+, \qquad \lim_{(Lx-k)\to 0^-} L(Ax + br_2) > 0 \quad \text{for} \quad x \in \mathbb{R}^n_+. \tag{5}$$

k is deduced from the straight line segment joining the fixed points:

$$x = \lambda\bar{x}_1 + (1-\lambda)\bar{x}_2, \quad \text{for} \quad \lambda \in (0, 1),$$

if $\lambda \in (0, 1)$ then the hyperplane $Lx - k = 0$ crosses the fixed point $\bar{x}$, thus

$$k = L\bar{x}. \tag{6}$$

Consider the sliding condition (5), the values of r_1, r_2, $\bar{x}$, and L, and the discontinuous control

$$u = \begin{cases} r_1 & \text{if} \quad Lx - k > 0 \\ r_2 & \text{if} \quad Lx - k < 0, \end{cases} \tag{7}$$

it is straightforward to show that any solution $x(t)$ with initial condition outside of the hyperplane $Lx = k$ reaches it in a finite time. It is known that the discontinuous control (7) takes values at the bounds of the restriction interval $[r_1, r_2]$, and it minimizes the arriving time to the hyperplane $Lx = k$ [8]. Now, according to the condition (5) and considering the Lyapunov function $V = \frac{1}{2}(Lx - k)^2$, the control (7) is the solution of the optimization problem consisting in finding the minimum of $\frac{dV}{dt}$ restricted to $u \in [r_1, r_2]$.

Once the sliding condition (5) is satisfied, a dynamic invariant is generated on the hyperplane $Lx = k$, this dynamic corresponds to the application of the called equivalent control u_{eq}, which is defined for x such that $Lx = k$, and it is derived from $L\dot{x} = 0$. Thus

$$L(Ax + bu_{eq}) = 0,$$

and

$$u_{eq} = -\frac{LAx}{Lb}.$$

With this result a global stabilizing positive control, for all $x \in \mathbb{R}^n_+$ is proposed:

$$u = \begin{cases} r_1 & \text{if} \quad Lx - k > 0 \\ -\dfrac{LAx}{Lb} & \text{if} \quad Lx - k = 0 \\ r_2 & \text{if} \quad Lx - k < 0. \end{cases} \tag{8}$$

3. Existence and design of a family of sliding hyperplanes

Consider a Metzler matrix A given by (1), a row-vector L, and a column-vector p, both in R^n, such that

$$L^T = \left(-A^{-1}\right)^T p.$$

Proposition 1. *Consider the feedback control system (1). If $p \in \mathrm{int}(\mathbb{R}^n_+)$, then there exists a sliding over the hyperplane $S(x) = \{x \in \mathbb{R}^n_+ \mid Lx - k = 0\}$.*

Proof. If $p \in \mathbb{R}^n_+$ a row-vector $L = -p^TA^{-1}$ can be deduced, such that

$$u_{eq} = -\frac{LAx}{Lb} > 0, \tag{9}$$

then

$$u_{eq} = \frac{p^Tx}{p^T(-A^{-1})b} > 0 \quad \text{for} \quad x \in \mathbb{R}^n_+.$$

From the Theorem 1 in [9] and choosing $r_2 > 0$ large enough ($0 < u_{eq} < r_2$), there exists a sliding on the hyperplane $S(x)$.□

Note that $A_{eq} = A + b\left(\frac{p^T}{p^T(-A^{-1})b}\right)$ is a Metzler matrix, since it is the sum of a Metzler matrix and a matrix with nonnegative entries.