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Optimal design of fractional order linear system with stochastic inputs/parametric uncertainties by hybrid spectral method

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ABSTRACT

This paper reports the design of a fractional linear system under stochastic inputs/uncertainties. The design methods were based on the hybrid spectral method for expanding the system signals over orthogonal functions. The use of the hybrid spectral method led to algebraic relationships between the first and second order stochastic moments of the input and output of a system. The spectral method could obtain a highly accurate solution with less computational demand than the traditional Monte Carlo method. Based on the hybrid spectral framework, the optimal design was elaborated by minimizing the suitably defined constrained-optimization problem.

1. Introduction

In recent years, fractional calculus has attracted considerable attention and is becoming increasingly popular because of its practical applications in a range of science and engineering fields [1–5]. This is because mathematical models based on fractional derivatives can describe a variety of natural phenomena, such as flexible structures [6], anomalous system [7], and viscoelastic materials [8]. On the other hand, fractional order systems are often studied using models with fixed deterministic parameters and inputs. Moreover, the input/parameters of these models are uncertain due to the inherent variability and/or incomplete knowledge. Therefore, the development of methods capable of designing fractional order systems with uncertainties is necessary.

The well-known Monte Carlo (MC) method is a typical approach for simulating stochastic models [9,10]. This method involves the generation of independent realizations of random inputs based on their prescribed probability distribution. For each realization, the data is fixed and the problem becomes deterministic. Solving the multiple deterministic realizations builds an ensemble of solutions, i.e. the realization of random solutions, from which statistical information can be extracted, e.g. mean and variance. This approach is simple to apply, involving only repeated deterministic simulations, but the convergence is slow and large numbers of calculations are required. For example, the mean values typically converge as $1/\sqrt{M}$, where M is the number of realizations.

Generalized polynomial chaos (gPC) [11–13] is a more recent approach for quantifying the uncertainty within system models. On the other hand, to simulate stochastic systems using the gPC method, the random inputs of many systems involve random processes approximated by truncated Karhunen–Loeve (KL) expansion, and the dimensionality of the input depends on the correlation lengths of these processes. For an input with a low correlation length (ideal white noise), the number of dimensions required for an accurate representation can be large, which increases the computational demand substantially using the gPC method.

A recent study [14] introduced a hybrid spectral method for quantifying the uncertainties in single input single output (SISO) fractional order systems. This paper extends the framework in Ref. [14] for the optimal design of a fractional SISO system under stochastic input/parametric uncertainty.

This paper is organized as follows: Section 2 briefly introduces a hybrid spectral method for uncertainty quantification in fractional order systems. Section 3 defines the suitable performance objectives coupled with the spectral method for the design of a stochastic linear fractional system. Section 4 considers examples ranging from integer to fractional order to demonstrate the proposed method.

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2. Fractional order system

This section summarizes the main concepts, definitions and basic results from fractional calculus, which are useful for further developments.

2.1. Governing equation for system dynamics

Among the many formulations of the generalized derivative with non-integer order, the Riemann–Liouville definition is used most commonly [15]

$$D_0^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_0^t \frac{f(\tau)}{(t-\tau)^{1-(m-\alpha)}} d\tau, \quad (1)$$

where $\Gamma(x)$ denotes the gamma function; m is the integer satisfying $m-1<\alpha<m$.

The Riemann–Liouville fractional integral of a function $f(t)$ is defined as

$$I_0^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau. \quad (2)$$

The Laplace transform for a fractional order derivative under zero initial conditions is defined as

$$L\{D_0^{\alpha} f(t)\} = s^{\alpha} F(s), \quad (3)$$

where $F(s)$ is the Laplace transform of $f(t)$.

Therefore, a fractional order single input single output (SISO) system can be described by the following fractional order differential equation

$$a_0 D_0^{\alpha_0} y(t) + a_1 D_0^{\alpha_1} y(t) + \cdots + a_l D_0^{\alpha_l} y(t) = b_0 D_0^{\beta_0} u(t) + b_1 D_0^{\beta_1} u(t) + \cdots + b_m D_0^{\beta_m} u(t), \quad (4)$$

or by the transfer function,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + \cdots + b_0 s^{\beta_0}}{a_l s^{\alpha_l} + \cdots + a_0 s^{\alpha_0}}, \quad (5)$$

where α_i and β_i are the arbitrary real positive numbers, and $u(t)$ and $y(t)$ are the system's input and output, respectively.

2.2. Operational matrices of block pulse function for analysis of fractional order system

Block pulse functions are a complete set of orthogonal functions that are defined over the time interval, $[0, \tau]$,

$$\psi_i = \begin{cases} 1 & \frac{i-1}{N}\tau \le t \le \frac{i}{N}\tau \\ 0 & \text{elsewhere} \end{cases}. \quad (6)$$

where N is the number of block pulse functions.

Therefore, any function that can be absolutely integrated on the time interval $[0, \tau]$ can be expanded into a series from the block pulse basis:

$$f(t) = \psi_N^T(t) C_f = \sum_{i=1}^{N} c_{f_i} \psi_i(t), \quad (7)$$

where $\boldsymbol{\psi}_N^T(t) = [\psi_1(t), \ldots, \psi_N(t)]$ constitutes of the block pulse basis. From here, the subscript, N, of $\boldsymbol{\psi}_N^T(t)$ is dropped out for the convenience of notation.

The expansion coefficients (or spectral characteristics) can be calculated as follows

$$c_{f_i} = \frac{N}{\tau} \int_{[(i-1)/N]\tau}^{(i/N)\tau} f(t)\psi_i(t) dt. \quad (8)$$

Furthermore, any function $g(t_1, t_2)$ absolutely integrable over the time interval, $[0, \tau] \times [0, \tau]$, can be expanded as

$$g(t_1, t_2) = \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}\psi_i(t_1)\psi_j(t_2) = \boldsymbol{\psi}^T(t_1) C_g \psi(t_2). \quad (9)$$

with expansion coefficients (or spectral characteristics) of

$$c_{ij} = \left(\frac{N}{\tau}\right)^2 \int_{[(i-1)/N]\tau}^{(i/N)\tau} \int_{[(i-1)/N]\tau}^{(i/N)\tau} g(t_1, t_2)\psi_i(t_1)\psi_j(t_2) dt_1 dt_2. \quad (10)$$

Eq. (2) can be expressed in terms of the operational matrix [16],

$$I_0^{\alpha} f(t) = \psi(t)^T A_{\alpha} C_f. \quad (11)$$

where the generalized operational matrix integration of the block pulse function, A_{α}, is

$$A_{\alpha} = P_{\alpha}^T = \left(\frac{\tau}{N}\right)^{\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{pmatrix} f_1 & f_2 & f_3 & \cdots & f_N \\ 0 & f_1 & f_2 & \cdots & f_{N-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & f_1 \end{pmatrix}^T. \quad (12)$$

The elements of the generalized operational matrix integration can be expressed as

$$f_1 = 1; \quad f_p = p^{\alpha+1} - 2(p-1)^{\alpha+1} + (p-2)^{\alpha+1} \quad \text{for} \quad p = 2, 3\ldots \quad (13)$$

The generalized operational matrix of a derivative of order α is

$$B_{\alpha} A_{\alpha} = I, \quad (14)$$

where I is the identity matrix.

The generalized operational matrix of the derivative can be used to approximate Eq. (1) as follows:

$$D_0^{\alpha} f(t) = \psi(t)^T B_{\alpha} C_f. \quad (15)$$

Using the operational matrix of the fractional order derivative, Eq. (4) can be rewritten in the following form:

$$A_G = (a_l D_{\alpha_l} + \cdots + a_0 D_{\alpha_0})^{-1}(b_m D_{\beta_m} + \cdots + b_o D_{\beta_o}). \quad (16)$$

The input and output of the system is thus linked by

$$C_Y = A_G C_U; \quad Y(t) = (C_Y)^T \boldsymbol{\psi}(t); \quad U(t) = (C_U)^T \boldsymbol{\psi}(t). \quad (17)$$

Closed-loop control systems normally comprise several elements, such as the controller and plant in Figs. 1 and 2, in terms of the transfer function and operational matrix, respectively. The operational matrix of a closed-loop system can be found using block diagram algebra similar to the block algebra used for the transfer function [10].

More on detail on the operational matrix with respect to the different polynomial functions can be found in [16,17] and the references therein.