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Exhaustive fluid vacation model with positive fluid rate during service ☆

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ABSTRACT

Using an elegant transform domain iterative operator approach exhaustive fluid vacation models with strictly negative fluid rate during service have been analyzed, recently. Unfortunately, the potential presence of positive fluid rate during service (when the fluid input rate is larger than the fluid service rate) inhibits the use of all previously applied methodologies and makes the extension of the analysis approaches of discrete vacation and polling models towards fluid vacation and polling models rather difficult.

Based on the level crossing analysis of Markov fluid models the paper introduces an analysis approach which is applicable for the stationary analysis of the fluid level distribution and its moments. In the course of the analysis compact new matrix exponential expressions are obtained for the distribution of fluid level during a busy period starting from a given positive fluid level and by exploiting the relations of upward and downward measures of fluid processes several matrix transformations are applied to avoid Kronecker expansion matrix multiplications.

Finally, the obtained fluid level distribution is related with previous results of fluid vacation models.

1. Introduction

Vacation and polling models with discrete (integer) customers have been studied [1,2] and applied on a wide range of application fields of stochastic modeling for long time [3,4]. An interesting property of the vacation models is the stochastic decomposition property [5], which means that the steady-state number of customers in the system can be decomposed to the sum of two independent random variables, the steady-state number of customers in the corresponding queue and the steady-state number of customers present in the system at arbitrary epoch in the vacation period.

With the evolution of fluid queueing models [6–9] and their use in applied modeling [10] the question of the fluid counterpart of vacation and polling models has arisen. A first step towards this direction is by Czerniak and Yechiali [11] whose model is rather limited with respect to the stochastic evolution of the considered process: the load and the fluid service rate of stations are constant and the only stochastic ingredient of the model is the switchover time.

A next step towards general fluid polling models was the analysis of fluid vacation models in which the fluid source is modulated by a background Markov chain, the service discipline is gated and the vacation (switchover) time is generally

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distributed [12]. A consecutive work evaluates the same model with exhaustive discipline under the assumption that fluid rate is strictly negative during the service period [13]. The analysis of these two fluid vacation models is based on an analysis approach which is inherited from discrete vacation model analysis. This analysis method starts with the transform domain description of the buffer content at service start epoch as a function of the buffer content at vacation start epoch and the same analysis at vacation start as a function of fluid level at service start. The solution of the obtained set of transform function equations gives the stationary distributions at these (vacation and service start) embedded time points. The final step of the analysis is the evaluation of the time stationary distribution of the fluid content based on the embedded ones.

The seemingly minor technical detail, strictly negative fluid rate during service, has a crucial role in the applicability of the outlined analysis approach. Our goal in this paper is to relax this small technical restriction. Unfortunately, the potential presence of positive fluid rate during service requires the introduction of a completely different analysis approach. Our approach is based on the analysis of level crossings during a vacation+service cycle. The analysis of level crossings in stochastic fluid models is introduced by the matrix analytic method community [8,9,14]. We apply several existing level crossing results from those works, but the analysis of fluid vacation model requires the introduction and analysis of previously not considered level crossing measures. For example, we obtained matrix exponential expressions for the mean number of level crossings during a busy period starting from a positive fluid level. A submitted extended version of [13] discusses the special case of fluid vacation models with exhaustive discipline and phase type distributed vacation time. It presents an analysis method based on the Kronecker sum of the background Markov chain generator and the phase type generator, but that methodology is not applicable when the vacation time is not phase type distributed.

Our long term research goal is the analysis of fluid polling models. The current paper is a small step towards this direction, where we still focus on vacation models (single station polling system), but relax an impractical restriction on the fluid rates. The rest of the paper is organized as follows. The next section presents the considered model behavior and modeling assumptions. The model behavior during service period is investigated in Section 3. The Laplace transform description of the stationary distribution of the fluid level and its mean are obtained in Sections 4 and 5, respectively. A small numerical example concludes the paper in Section 6.

2. Model description

We consider a fluid vacation model with Markov modulated load and exhaustive discipline. The model has an infinite fluid buffer.

The input fluid flow of the buffer is determined by a modulating CTMC ($\Omega(t)$ for $t \geq 0$) with state space $\mathcal{S} = \{1, \ldots, N\}$ and generator $\mathbf{Q}$. When this Markov chain is in state j ($\Omega(t) = j$) then fluid flows to the buffer at rate r_j for $j \in \{1, \ldots, N\}$. We define the diagonal matrix $\mathbf{R} = \text{diag}\langle r_1, \ldots, r_N \rangle$. During the service period the server removes fluid from the buffer at finite rate $d > 0$. Consequently, when the overall Markov chain is in state j ($\Omega(t) = j$) then the fluid level of the buffer during the service period changes at rate $r_j - d$, otherwise during the vacation periods it changes at rate r_j, because there is no service.

It is an important restriction in the current work that we consider only non-zero fluid rates, that is, $r_j > 0$ and $r_j - d \neq 0$ for $j \in \{1, \ldots, N\}$. The difficulty of relaxing these restrictions is similar to the generalization of Markov fluid models without zero rates to the ones with possible zero fluid rates, but the notations used for the analysis of the current problem without zero fluid rates are rather complicated and we omit the case with zero fluid rate which requires a more complex set of notations.

We subdivide the set of states into $\mathcal{S}^+$ and $\mathcal{S}^-$ according to the sign of $r_i - d$. Without loss of generality we assume that the indexes of the states in $\mathcal{S}^+$ are lower than the ones in $\mathcal{S}^-$. Accordingly, matrices $\mathbf{Q}$ and $\mathbf{R}$ are partitioned as

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}^{++} & \mathbf{Q}^{+-} \\ \mathbf{Q}^{-+} & \mathbf{Q}^{--} \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} \mathbf{R}^{+} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^{-} \end{bmatrix}. \tag{1}$$

When the fluid rate is strictly negative during service, which is the assumption in [13], then $\mathcal{S}^+ = \emptyset$, and all below discussed analytical problems associated with the $\mathcal{S}^+$, $\mathcal{S}^-$ division of the states are avoided.

In the vacation model the length of the service period is determined by the applied discipline. In this work we consider the exhaustive discipline. Under exhaustive discipline the fluid is removed during the service period until the buffer becomes empty. Each time the buffer becomes empty the server takes a vacation period. During vacation periods there is no service thus the fluid level of the buffer is increasing by the actual fluid rates. The consecutive vacation times are independent and identically distributed (i.i.d.). The random variable of the vacation time, its probability distribution function (pdf), its Laplace transform (LT), its ith moment and its squared coefficient of variation are denoted by σ, $\sigma(t) = \frac{\mathrm{d}}{\mathrm{d}t}Pr(\sigma < t)$, $\sigma^*(s) = E(e^{-s\sigma})$, $E(\sigma^i)$ and $c_\sigma^2 = \frac{E(\sigma^2)}{E^2(\sigma)} - 1$, respectively. We define the *cycle* time (or simple cycle) as the time between just after the starts of two consecutive service periods. In the sequel we apply the notation $\sigma^*(s) = E(e^{-s\sigma}) = \int_t e^{-st}\sigma(t)\mathrm{d}t$ not only for scalar values but for square matrices as well. E.g. for matrix $\mathbf{X}$ we have $\sigma^*(\mathbf{X}) = E(e^{-\mathbf{X}\sigma}) = \int_t e^{-\mathbf{X}t}\sigma(t)\mathrm{d}t$.

We set the following assumptions on the fluid vacation model:

- A.1 The generator matrix $\mathbf{Q}$ of the modulating CTMC is finite and irreducible.
- A.2 The fluid rates are positive and finite, i.e. $r_j > 0$ for $j \in \{1, \ldots, N\}$.