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Discontinuous Galerkin Isogeometric Analysis for the biharmonic equation

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ABSTRACT

We present and analyze an interior penalty discontinuous Galerkin Isogeometric Analysis (dG-IgA) method for the biharmonic equation in computational domain in $\mathbb{R}^d$ with $d = 2, 3$. The computational domain consists of several non-overlapping sub-domains or patches. We construct B-Spline approximation spaces which are discontinuous across patch interfaces. We present *a priori* error estimate in a discrete norm and numerical experiments to confirm the theory.

1. Introduction

In this paper, we consider the fourth-order Dirichlet boundary value problem: find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\Delta^2 u = f \quad \text{in } \Omega, \qquad u = g_0, \quad \text{and} \quad \mathbf{n} \cdot \nabla u = g_1 \quad \text{on} \quad \partial\Omega, \tag{1}$$

where $\mathbf{n}$ is the external unit normal vector to the boundary $\partial\Omega$, the bi-Laplacian operator $\Delta^2 := \Delta\Delta$ with Δ as the Laplace operator, f is a given source function, g_0, g_1 are boundary data and $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is a bounded Lipschitz computational domain with the boundary $\partial\Omega$. We assume that the domain Ω is generated by Computer Aided Design (CAD) system and represented by a single or multiple patches which are images of the parameter domain $(0, 1)^d$ by spline or NURBS maps.

The model problem (1) is an example of a fourth-order elliptic problem occurring usually in various model of computational mechanics such as the Bernoulli–Euler beam and the Poisson–Kirchhoff thin plate theories [1,2]. Several numerical solution techniques for the fourth-order problem have been studied including conforming and non-conforming finite element methods (FEM) and mixed finite element methods see, e.g. [3,4]. The construction of conforming methods for such problems require finite element spaces of $H^2(\Omega)$. Such H^2-conforming methods are known to require continuously differentiable (i.e. C^1-) piece-wise polynomials on the elements. This is however known to be considerably difficult to construct practically. Examples of such conforming finite elements for such a problem are the Argyris element which uses polynomials of degree $p = 5$ for triangular elements, the reduced Hsieh–Clough–Tocher (rHCT) or Hsieh–Clough–Tocher (HCT) element also called macro-elements, which uses cubic polynomials for sub-partition triangular elements and the Bogner–Fox–Schmit element which uses bi-cubic functions for rectangular elements . For the non-conforming finite element, a typical example for solving such a model problem is the Morley element which uses piece-wise quadratic polynomials, see e.g. [3].

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Alternatively, the fourth-order partial differential equation (PDE) can be solved by using the interior penalty discontinuous Galerkin finite element methods. The interior penalty methods date back to [5] where Douglas and Dupont combined conforming continuous finite element with penalty terms which led to consistent schemes to derive *a priori* error estimates. In [6], the continuous Galerkin (cG), discontinuous Galerkin (dG) and stabilization techniques combined in solving the fourth-order elliptic problems and applied to thin plate bending theory problems in structural mechanics and to a strain gradient theory problem. The continuous/discontinuous Galerkin method has further been applied to the biharmonic problem on closed surfaces [7]. A continuous interior penalty *hp*-version of the interior penalty discontinuous Galerkin finite element method for fourth-order elliptic problems has also been studied, see, e.g., [8,9]. Finally, we mention that a continuous interior penalty method for fourth order elliptic boundary value problems including Kirchhoff plates on polygonal domains has been analyzed in [10–12]

In most recent times, isogeometric analysis (IgA) has been proposed to approximate solutions of PDEs, see, e.g. [13]. The IgA uses the same class of basis functions for both representing the geometry of the domain and approximating the solution of the PDEs. Furthermore, the IgA has $(p-1)$-continuous differentiable basis i.e. $C^{(p-1)}$ with degree $p \geq 1$ functions which makes it an ideal scheme for the approximation of higher order PDEs including the biharmonic problem (1), see, e.g. [14].

In this paper, we will present *a priori* error estimate for multi-patch interior penalty discontinuous Galerkin isogeometric analysis (dG-IgA) for biharmonic equation on conforming patches with matching meshes. The dG-IgA or Nitsche coupling method has been introduced and analyzed for second order elliptic problems, see e.g., [15–19]. Following the monograph of Di Pietro and Ern [20], our analysis will require three main ingredients namely; discrete stability, consistency and boundedness of the discrete bilinear form. Using approximation estimates for h-refined IgA meshes from [13] and [14], we will then present *a priori* error estimate in an appropriate discrete norm.

The rest of the paper is organized as follows. In Section 2, we introduce function spaces, weak formulation and the isogeometric analysis framework. Section 3 involves the derivation of the interior penalty discontinuous Galerkin scheme. Then, in Section 4, we introduce a discrete NURBS space V_h and a discrete norm $\|\cdot\|_h$ and prove the coercivity of the bilinear form. The boundedness of the bilinear form is asserted in a product space $V_{h,*} \times V_h$, where we will need another discrete norm $\|\cdot\|_{h,*}$ defined on the vector space $V_{h,*}$. The error analysis of the dG-IgA scheme is presented in Section 5. In Section 6, we present and discuss numerical experiments to confirm our theoretical results. Finally, we draw some conclusions and discuss future works in Section 7.

2. Preliminaries

Let Ω be a bounded Lipschitz domain with boundary $\partial\Omega$. We introduce the Sobolev space $H^s(\Omega) := \{v \in L_2(\Omega) : D^\alpha v \in L_2(\Omega), \text{ for } 0 \leq |\alpha| \leq s\}$, where $L_2(\Omega)$ denote the space of square integrable functions and let $\alpha = (\alpha_1, \dots, \alpha_d)$ be a multi-index with non-negative integers $\alpha_1, \dots, \alpha_d$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $D^\alpha := \partial^{|\alpha|}/\partial x^\alpha$, and associate with the Sobolev space $H^s(\Omega)$ the norm $\|v\|_{H^s(\Omega)} = \left(\sum_{0\leq|\alpha|\leq s} \|D^\alpha v\|^2_{L_2(\Omega)}\right)^{1/2}$ see, e.g. [21].

The variational formulation of the biharmonic problem (1) reads: find $u \in V_D$ such that

$$a(u, v) = \ell(v), \quad \forall v \in V_0, \tag{2}$$

where the bilinear and linear forms are given by

$$a(u, v) = \int_\Omega \Delta u \Delta v \, dx \quad \text{and} \quad \ell(v) = \int_\Omega f v \, dx, \tag{3}$$

and the hyperplane and test space given by $V_D := \{v \in H^2(\Omega) : v = g_0, \quad \mathbf{n}\cdot\nabla v = g_1 \quad \text{on} \quad \partial\Omega\}$ and $V_0 := \{v \in H^2(\Omega) : v = 0, \quad \mathbf{n}\cdot\nabla v = 0 \quad \text{on} \quad \partial\Omega\}$. The existence and uniqueness of the variational problem (2) follows the well-known Lax–Milgram lemma see e.g. [3].

2.1. B-spline and isogeometric analysis

We refer the reader to [22] for detailed study on B-splines or NURBS based Galerkin methods. For the unit interval $\widehat{\Omega} = [0, 1]$, we define a vector $\Xi = \{0 = \xi_1, \dots, \xi_{n+p+1} = 1\}$ with a non-decreasing sequence of real numbers in the parameter domain $\widehat{\Omega} = [0, 1]$ called a knot vector. Given $\Xi, p \geq 1$, and n the number of basis functions, the univariate B-spline basis functions are defined by the Cox–de Boor recursion formula as follows:

$$\widehat{B}_{i,0}(\xi) = \begin{cases} 1 & \text{if} \quad \xi_i \leq \xi < \xi_{i+1}, \\ 0 & \text{else}, \end{cases}$$
$$\widehat{B}_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i}\widehat{B}_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}}\widehat{B}_{i+1,p-1}(\xi), \tag{4}$$

where a division by zero is defined to be zero. We note that a basis function of degree p is $(p-m)$ times continuously differentiable across a knot value with the multiplicity m. If all internal knots have the multiplicity $m = 1$, then B-splines of degree p are globally $(p-1)$-continuously differentiable.