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On-diagonal lower estimate of heat kernels for locally finite graphs and its application to the semilinear heat equations

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ABSTRACT

In this paper we establish a new on-diagonal lower estimate of heat kernels for connected, weighted, locally finite graphs. The result is then used to deal with the nonexistence of global solutions for a semilinear heat equation on locally finite graphs. Our results provide remarkable improvements to the work that was done recently by Lin and Wu (2017) [8,18].

1. Introduction

As is known to us, many structures in our real life can be represented by a connected graph whose vertices represent nodes, and whose edges represent their links, such as the internet, brain, organizations, and so on. In recent years, the study of partial differential equations on graphs is a very attractive research topic (see, e.g., [1–9] and references cited therein). This work has made significant complements to the researches of application-oriented partial differential equations. As a useful tool, the estimates of heat kernels on graphs have also stimulated the interest of many researchers (see, e.g., [10–19]).

Recently, Lin and Wu [18] established an on-diagonal lower estimate of continuous-time heat kernels on locally finite graphs under polynomial volume growth condition, as follows:

Proposition 1.1 (*[18], Theorem 1.1*)**.** *Suppose that, for all $x \in V$ and $r \geq r_0$, the polynomial volume growth $V(x, r) \leq c_0 r^m$ holds, where r_0, c_0, m are positive constants. Then, for all large enough t and all $x \in V$,*

$$p(t, x, x) \geq \frac{1}{4\mathcal{V}(x, Ct \log t)}, \tag{1.1}$$

where $C > 2e\left(\sqrt{D_\mu} \vee 1\right)$.

In [8], Lin and Wu used the above on-diagonal lower estimate to investigate the existence and nonexistence of global solutions for the following semilinear heat equation on locally finite connected weighted graphs:

$$\begin{cases} u_t = \Delta u + u^{1+\alpha}, & (t, x) \in (0, +\infty) \times V, \\ u(0, x) = a(x), & x \in V, \end{cases} \tag{1.2}$$

where α is a positive parameter, $a(x)$ is bounded, non-negative and non-trivial in V. They proved the following result:

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Proposition 1.2 *([8], Theorem 3.1).* *Suppose that, for all $x \in V$ and $r \geq r_0$, the polynomial volume growth $V(x, r) \leq c_0 r^m$ holds, where r_0, c_0, m are positive constants. If $0 < m\alpha < 1$, then there is no non-negative global solution of (1.2) in $[0, +\infty)$ for any bounded, non-negative and non-trivial given initial value.*

It should be noted that the above-mentioned study for semilinear heat equation (1.2) is motivated by a previous work of Fujita [20] on Euclidean spaces. In [20], Fujita showed that, if $0 < m\alpha < 2$, then there does not exist a non-negative global solution of Eq. (1.2) for any non-trivial non-negative initial data. This prompts us to ask a natural question: Is there a similar conclusion for Eq. (1.2) on locally finite graphs? Theorem 1.5 gives an affirmative answer to this question.

In order to prove that the assertion of Proposition 1.2 is valid in a more general setting for $0 < m\alpha < 2$, we first establish a more accurate on-diagonal heat kernel lower estimate than the estimate (1.1) for locally finite graphs, as follows:

Theorem 1.3. *Let (G, ω, μ) be a locally finite, connected, weighted graph. Suppose that, for all $x \in V$ and $r \geq r_0$,*

$$\mathcal{V}(x, r) \leq c_0 r^m,$$

where r_0, c_0, m are some positive constants. Then, for all large enough t and all $x \in V$,

$$p(t, x, x) \geq \frac{1}{4\mathcal{V}(x, C\sqrt{t}\log t)}, \tag{1.3}$$

where $C > \frac{m}{2}$.

Remark 1.4. It is easy to observe that $C_2\sqrt{t}\log t \leq C_1 t\log t$ holds for large enough t, which implies $\mathcal{V}(x, C_2\sqrt{t}\log t) \leq \mathcal{V}(x, C_1 t\log t)$ in view of the definition of $\mathcal{V}(x, r)$ in Section 2. This means that the heat kernel estimate (1.3) is more accurate than that given in (1.1).

Further, with the help of heat kernel estimate (1.3), we can generalize Proposition 1.2 by replacing the setting condition $0 < m\alpha < 1$ with a more general condition $0 < m\alpha < 2$. Namely, we have

Theorem 1.5. *Let (G, ω, μ) be a locally finite, connected, weighted graph. Suppose that, for all $x \in V$ and $r \geq r_0$, the polynomial volume growth $\mathcal{V}(x, r) \leq c_0 r^m$ holds, where r_0, c_0, m are positive constants. If $0 < m\alpha < 2$, then there is no non-negative global solution of (1.2) in $[0, +\infty)$ for any bounded, non-negative and non-trivial given initial value.*

The paper is organized as follows. In Section 2, we introduce some definitions and notations on graphs, we also review some known results about heat kernels on graphs. In Sections 3 and 4, we give the proofs of Theorems 1.3 and 1.5 respectively.

2. Preliminaries

2.1. Weighted graphs

Let $G = (V, E)$ be a locally finite, connected graph. Here V is the vertex set and E is the edge set that can be viewed as a symmetric subset of $V \times V$. For $(x, y) \in E$, we write $x \sim y$ for short. Let $\omega : V \times V \to [0, \infty)$ be an edge weight function that satisfies $\omega_{xy} = \omega_{yx}$ for all $x, y \in V$ and $\omega_{xy} > 0$ if and only if $x \sim y$. Let $\mu : V \to (0, \infty)$ be a positive measure on V and satisfy $\mu_0 := \inf_{x\in V}\mu(x) > 0$. The triplet (G, ω, μ) is called a weighted graph, where $G = (V, E)$ is a combinatorial graph.

In this paper, all the graphs in our concerns are assumed to satisfy

$$D_\mu := \sup_{x\in V}\frac{m(x)}{\mu(x)} < \infty,$$

where $m(x) = \sum_{y\sim x}\omega_{xy}$.

The connected graph structure induces the graph distance $d(x, y)$, which is the smallest number of edges of a path connecting x and y. For a weighted graph (G, ω, μ) and the graph distance $d(\cdot, \cdot)$, we denote the balls by

$$B(x, r) = \{y \in V : d(x, y) \leq r\},$$

where $r \geq 0$. The volume of balls $B(x, r)$ is written as $\mathcal{V}(B(x, r))$, or simply $\mathcal{V}(x, r)$, which is defined by

$$\mathcal{V}(x, r) = \sum_{y\in B(x,r)}\mu(y).$$

Moreover, a graph (G, ω, μ) has polynomial volume growth of degree $m > 0$, if there exist constants $c_0 > 0$ and $r_0 > 0$, such that for all $x \in V$, $r \geq r_0$,

$$\mathcal{V}(x, r) \leq c_0 r^m.$$