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Combination of the meshless finite difference approach with the Monte Carlo random walk technique for solution of elliptic problems

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ABSTRACT

This paper proposes a stochastic approach for the fast and effective numerical analysis of the second order elliptic differential equations. It is based upon the well-known Monte Carlo (MC) method with a random walk (RW) technique, carried out on the grid of points. This method allows for accurate estimation of the solution of the differential equation at selected point(s) of the domain and/or its boundary. It extends the standard formulation of the Monte Carlo–random walk (MC–RW) approach by means of its appropriate combination with the meshless version of the finite difference method. In this manner, the proposed approach may deal with elliptic equations in more general non-homogeneous form as well as boundary conditions of both essential and natural types. Moreover, arbitrarily irregular clouds of nodes may be used, with no a-priori imposed nodes structure. Therefore, the meshless MC/RW approach may be applied to the significantly wider class of problems with more complex geometry.

This concept was examined on variety of 2D boundary value problems. Selected numerical results are presented and discussed. A simple Matlab code is included as well.

1. Introduction

Problems of mechanics and civil engineering are modeled by considering a variety of appropriate simplifying assumptions. The resulting mechanical and mathematical models may be of deterministic or probabilistic nature. In deterministic models, a particular state (either continuous or discrete) is uniquely assigned to input data and no element of randomness occurs. This means that the evolution of a system in a deterministic model is predetermined and depends solely on the initial parameters or their former values. The deterministic model is a useful and most commonly applied model in the description of many physical, mechanical, biological or economic phenomena. It is also an important tool in the mathematical optimization process. In most cases, it leads to a system of partial differential equations (PDE), constituting a boundary or initial–boundary value problem (BVP or IVP), investigated by means of numerical methods (named *deterministic* as well), since their exact analytical solution is possible to obtain in simple cases only. There exists a variety of deterministic numerical methods. Most of them require discretization of the problem domain and approximation of the unknown function. One may distinguish here finite difference method (FDM, [1,2]), possibly the oldest method for numerical analysis of the boundary value problems; finite element method (FEM, [3]), the most developed and mature method, constituting the base for majority of commercial and scientific computational codes as well as a wide group of meshless methods (MM, [4,1,5–7]). In these meshless methods, nodes are not connected by any a-priori imposed structure like finite element (in FEM), regular mesh (in

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FDM) or any mapping restrictions. Nodes may be arbitrarily irregularly scattered, forming an unstructured cloud of nodes. Therefore, all modifications of the nodes (nodes shifting, adding, removal) may be performed with no difficulties. However, any such method remains meshless as long as the approximation of the unknown function is built upon nodes and only nodes. Meshless methods differ from each other with respect to the manner of generating approximation schemes. Though their efficiency is quite varied, they are usually more convenient in such problems as the moving boundary, reduction of material, crack development, presence of concentrated forces, plate deflection, presence of holes and during adaptation process, in which a cloud of nodes is under current refinement. One of the oldest and therefore, the most developed meshless method is the meshless (generalized) finite difference method (MFDM, [8,1,7,9]), which uses moving weighted least squares (MWLS, [10,11,1]) approximation technique.

The probabilistic class of mathematical models specifies governing equations that combine one or more random variables with other non-random variables. Predominantly, the probability distribution functions are assigned to those random variables. Regardless of the applied method of analysis, one faces multiple deterministic problems, each representing various random states. Each time a solution method is performed for the entire set of problems and their particular solutions may be different from each other, due to the randomness of the initial data and/or initial conditions. Therefore, appropriate weighted averaging or other optimization processes have to be additionally applied in order to select *typical standard* or *the strongest* member (the optimal solution). Though probabilistic models require specially selected probabilistic methods of analysis, analytical and numerical methods from deterministic modeling may be required as well. The most common *probabilistic* approaches in mechanics (usually in combination with deterministic ones) include fuzzy sets (FS, [12]) analysis, in which fuzzy models are applied to the initial data, with a specified membership function, characterizing data randomness; genetic and evolutionary algorithms (GA and EA, [13]), inspired by a biological evolution and tailored for multidimensional optimization problems; artificial neural networks (ANN, [14]), inspired by a biological nervous system, being a representative for a wider class of machine learning domain as well as Monte Carlo (MC, [15–40]) methods. Their general idea is to perform a set of a random sampling as long as the properly defined success is achieved. Number of successes related to the total number of trials, scaled by a dimension quantity (area, function value, etc.) may estimate the exact solution of a problem, providing the number of trials is large enough.

Though MC preliminaries were known in the eighteenth century, the method owes its name, general formulation as well as the first mature applications (atomic bomb calculations for neutron transport) to Ulam and von Neumann [15,27]. Since then, the MC method became one of the fundamental tools of probabilistic type in numerical analysis of mathematical problems. Kac and Donsker used large deviation calculations to estimate eigenvalues of a linear Schrödinger equation [17], while Forsythe and Leibler derived a MC method for solving special linear systems related to discrete elliptic PDE problems [16]. Curtiss investigated MC methods for solving a system of algebraic equations and compared them with direct and iterative solvers [18,19]. Numerical integration by MC, including multidimensional cases, has been investigated for instance by Niederreiter [23] and Mascagni [29]. However, numerical analysis of BVP and IVP with MC methods constitutes the largest branch of their applications [20–22,24–26,28,30–32]. The pioneering works of Muller, Reynolds and Hoshino et al. [20–22] present a formulation of the Monte Carlo method with a random walk technique at regular grids of points. It allows to estimate the solution of a Laplace equation at the specified internal point by performing a set of random walks (trials) through other grid points until the boundary node with known solution value is reached (success). It has been proved [21] that this solution is convergent to the solution of a Laplace equation obtained by means of the finite difference method (FDM). However, a generation and solution of a system of difference equations, combining all unknown solution values, which is typical for the FDM, is not required here. This simple solution approach was used, modified and improved by many investigators in subsequent years. As it is impossible to mention all interesting works, a survey is limited to those which were the primary inspiration of this research. A continuous random walk procedure, minimizing the solution error has been proposed by Booth [25]. Self-adaptive, grid-free algorithm, with improved solution smoothness and its application to diffusion equations may be found in [26]. State-of-the-art as well as recent developments (up to date) are presented in [28,40]. An effective reduction of random walk steps has been suggested by Dimov et al. [31]. A weighted version of MC methods was considered by Mikhailov [32]. The MC treatment of complex 2D geometries has been investigated by Ramachandran et al. [35]. More recent works include, for instance, a reduction of the MC error by a new probabilistic sampling [36], a development of a stochastic limit theory and governing equations of a continuous time random walk [37,38], an improvement of the simulation efficiency for radiative transfer problems with strongly frequency-dependent opacities [39], as well as an analysis of heat conduction equations [40].

The main objective of this paper is to develop the original Monte Carlo with random walk (MC/RW) solution approach towards analysis of a wider class of 2D elliptic problems. This innovative concept assumes incorporation of selected discretization and approximation schemes, typical for meshless methods (MM), for the first time ever. Especially, features of a moving weighted least squares approximation (MWLS) are applied in order to reformulate standard random walk principles. In such a manner, the meshless version of the RC/RW algorithm, proposed here and being a combined probabilistic (MC, RW)–deterministic approach (MFDM, MWLS), permits

- analysis of Poisson equations, with non-homogeneous right-hand side functions and arbitrary essential boundary conditions,
- accounting for heterogeneous boundary conditions of natural type,
- application of arbitrarily irregular meshes and clouds of nodes (neither element structure not the mesh regularity is required),
- analysis of elliptic equations, with variable material functions.

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