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Nonconforming polynomial mixed finite element for the Brinkman problem over quadrilateral meshes

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ABSTRACT

This work provides a new mixed finite element method for the Brinkman problem over arbitrary convex quadrilateral meshes. The velocity is approximated by piecewise polynomial element space which is $H(\text{div})$-nonconforming, and the pressure is approximated by piecewise constant. We give the convergence analysis of our element, and especially show the robustness with respect to the Darcy limit. Moreover, via a discrete de Rham complex, a higher-order approximation error term is obtained for incompressible flow. Numerical examples verify our theoretical findings.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and $\partial\Omega$ be its boundary. We consider the following Brinkman problem of porous media flow: Find the velocity $\boldsymbol{u}$ and the pressure p satisfying

$$\begin{aligned} -\text{div}(\nu\nabla\boldsymbol{u}) + \alpha\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega, \\ \text{div}\,\boldsymbol{u} &= g \quad \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here $\nu > 0$ is the viscosity coefficient and $\alpha \geq 0$ is the dynamic viscosity divided by the permeability. The right-hand-side terms $\boldsymbol{f}$ and g are known forcing terms. This problem arises in many fields, such as underground water hydrology, the petroleum industry, the automotive industry, biomedical engineering, and heat pipes modeling. For the convenience of our mathematical analysis, we assume that the coefficients ν and α are constants. The known term g fulfills the solvability condition $\int_\Omega g\,\mathrm{d}\boldsymbol{x} = 0$.

Mixed finite element methods are a powerful tool in fluid mechanics. If α is small, this model behaves like a Stokes problem, thus classical Stokes elements seem to be natural choices. Unfortunately, if ν tends to zero, this model turns to satisfying Darcy's law for porous media flow. It has been shown in [1] that many classical Stokes elements are not robust in such a case. Conversely, many successful $H(\text{div})$ elements for Darcy problems also fail for the original problem (1.1). There are several ways of escaping this dilemma. In [2], the authors provided two frameworks based on the classical velocity–pressure formulation. The first method is to use H^1-conforming divergence-free Stokes elements, such as Scott–Vogelius's P_k-P_{k-1} elements for $k \geq 2$ [3–5]. Recent contributions on this kind of elements include [6–9], etc. The second method is to modify

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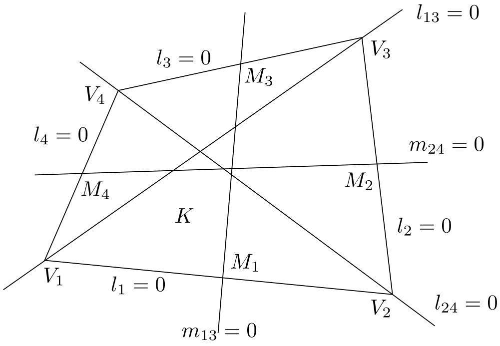

Fig. 1. A convex quadrilateral K and the related notations.

existing $H(\text{div})$ elements such as the Raviart–Thomas [10] and Brezzi–Douglas–Marini families [11] to impose some weak continuity of the tangential trace, see [1,2,12,13]. Indeed, they are all H^1-nonconforming but $H(\text{div})$-conforming. We remark that there are many other approaches without utilizing the original velocity–pressure formulation, although they are out of the range of this paper. They include various versions of stabilized methods [2,14–17], and the schemes by introducing new unknowns such as the vorticity [18–20]. Moreover, as variants of the standard finite element method, the weak Galerkin method [21,22] and virtual element method [23] are also promising for the Brinkman problem over complicated partitions.

We now focus on H^1-nonconforming finite element methods based on the classical velocity–pressure formulation. Most of the aforementioned elements of this type are $H(\text{div})$-conforming constructed over triangles [1,2,13]. An extension to the rectangular case can be found in [12]. Note that all these elements above adopt polynomial shape functions, which are simple to represent and easy to compute. However, on general quadrilaterals, it seems very difficult (even impossible) to construct a pure polynomial $H(\text{div})$-conforming element. Instead, the shape functions can be selected as rational functions in a parametric [24] or nonparametric manner [25]. Splines or composite methods [26] can also be utilized. As a consequence, polynomial finite elements will lead to an $H(\text{div})$-nonconforming approximation. For the Stokes problem, a successful nonconforming divergence-free element on general convex quadrilaterals, based on an incomplete quadratic polynomial space, was designed by Zhang [27] with a discrete Stokes complex. As for the problem (1.1), Zhang et al. [28] have introduced a rectangular finite element of 8 degrees of freedom (DoFs) which is also $H(\text{div})$-nonconforming. However, a direct extension to general quadrilaterals is still unknown.

In this work, a new mixed finite element method is proposed for the Brinkman problem (1.1) over arbitrary convex quadrilateral meshes. The velocity space consists of 12-DoF finite elements with piecewise polynomial shape functions, where the DoFs are the same as those in [12], but the global approximation is $H(\text{div})$-nonconforming. We adopt piecewise constant element for approximating the pressure. The convergence analysis is provided with an $O(h)$ convergence order, and the robustness with respect to the Darcy limit is also shown, where the uniform convergence order is $O(h^{1/2})$ under some mild conditions on the right-hand-side terms. Moreover, we construct a discrete de Rham complex with some commutative property. This hints a better approximation ability of our element for incompressible flow. Numerical examples are consistent with our theoretical findings. We also observe from the numerical results that, over uniform rectangular partitions, our method has a better performance than we expect.

The rest of this paper is organized as follows. We first introduce our polynomial quadrilateral finite element in Section 2. In Section 3, the convergence and uniform convergence analysis are provided. We next construct a discrete de Rham complex in Section 4. Finally, numerical examples are given in Section 5. Throughout the paper, for a domain D, $\boldsymbol{n}$ and $\boldsymbol{t}$ will be the unit outward normal and tangent vectors on ∂D, respectively. The notation $P_k(D)$ denotes the usual polynomial space over D of degree no more than k.

2. A new polynomial finite element on quadrilaterals

2.1. Notations of a quadrilateral and an auxiliary affine transformation

As in Fig. 1, let K be an arbitrary convex quadrilateral. The four vertices of K are given by V_1, V_2, V_3, V_4 in a counterclockwise order, and the ith edge of K is denoted by $E_i = V_iV_{i+1}$, whose equation is written as $l_i(x, y) = 0$, $i = 1, 2, 3, 4$. Here and throughout the paper, the index i is taken modulo four. For each E_i, M_i denotes its midpoint. The equations of lines through M_1M_3, M_2M_4, V_1V_3 and V_2V_4 read as $m_{13}(x, y) = 0$, $m_{24}(x, y) = 0$, $l_{13}(x, y) = 0$ and $l_{24}(x, y) = 0$, respectively.

In order to provide our new finite element, we shall use an auxiliary affine transformation for K introduced in [29]. Set $l_{13}^* = \lambda_{13}l_{13}$ and $l_{24}^* = \lambda_{24}l_{24}$, where the constants λ_{13} and λ_{24} are determined such that $l_{13}^*(V_4) = l_{24}^*(V_1) = 1$. If we take $h_1 = -l_{13}^*(V_2)$ and $h_2 = -l_{24}^*(V_3)$, then $h_1 > 0$ and $h_2 > 0$ if and only if K is convex. We now define a reference quadrilateral