ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Unfitted extended finite elements for composite grids

Luca Formaggia, Christian Vergara *, Stefano Zonca

MOX, Dipartimento di Matematica, Politecnico di Milano, Italy

ARTICLE INFO

Article history:
Received 13 July 2017
Received in revised form 14 May 2018
Accepted 22 May 2018
Available online xxxx

Keywords:
Unfitted meshes
Extended finite element method
Discontinuous Galerkin

ABSTRACT

We consider an Extended Finite Elements method to handle the case of composite independent unstructured grids that lead to unfitted meshes. In particular, we address the case of two overlapped meshes, a background and a foreground one, where the thickness of the latter is smaller than the elements of the background mesh. This situation may lead to elements split into several portions, thus generating polyhedral elements. We detail the corresponding discrete formulation for the Poisson problem with discontinuous coefficients. We also provide some technical details for the 3D implementation. Finally, we provide some numerical examples with the aim of showing the effectiveness of the proposed formulation.

1. Introduction

The need of solving elliptic problems with unfitted meshes may arise in many applicative contexts, e.g. when the coefficients are discontinuous across an internal interface or when different overlapping meshes are composed to manage coupled problems.

For the first class of problems, conforming/matching grids at the interface are difficult to generate, since the internal interface provides a severe constraint for the mesh generation. To overcome such a problem, in [1] an unfitted strategy has been proposed for Finite Elements, based on doubling the degree of freedoms in the elements cut by the interface (*eXtended Finite Element Method (XFEM)*), see also [2]. This allows one to use a mesh which is completely independent of the interface position, see also [3,4] for 3D applications, [5] for the case of a prescribed moving interface, [6,7] for the case of solid mechanics, [8,9] for the Stokes problem, [10] for the fluid–fluid coupling, and [11] for the fluid–structure interaction problem.

As regards composite meshes, usually we have to face a (background) mesh on which another (foreground) mesh is overlapped, to account for example for different physical properties or even physical laws. In such a case we have that the background mesh is again non-matching the interface given by the intersection between the two meshes. To handle this problem with Finite Elements, a new strategy has been proposed in [12], see [13] for 3D applications, [14,15] for the fluid equations, [16] for the fluid–structure interaction problem.

Both these situations lead to mesh elements with complex shape (polyhedra), and the proposed strategies guarantee to maintain the accuracy of the standard Finite Elements method. The common ingredient is the Discontinuous Galerkin (Nitsche) mortaring at the internal interface.

In this paper, starting from the two strategies mentioned above [1,12], we introduce an XFEM formulation adapted to the case of composite meshes. This allows us to treat situations where, due to the thin thickness of the foreground mesh, possibly smaller than the background characteristic mesh size, an element of the background mesh is split into two or more

* Corresponding author.
E-mail addresses: luca.formaggia@polimi.it (L. Formaggia), christian.vergara@polimi.it (C. Vergara), stefano.zonca@polimi.it (S. Zonca).

https://doi.org/10.1016/j.camwa.2018.05.028
0898-1221/

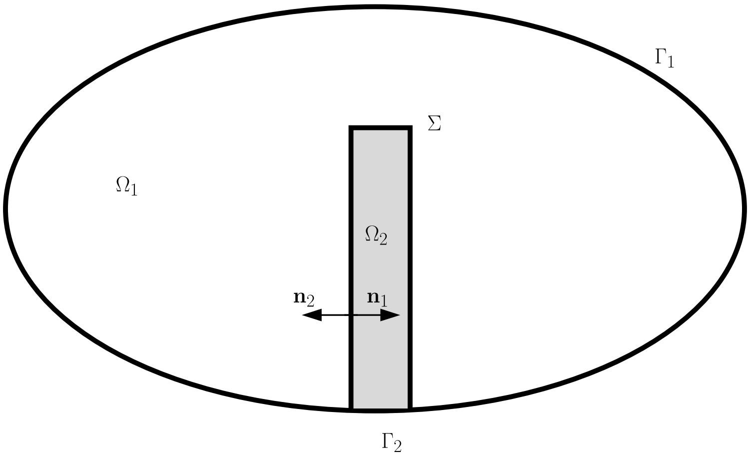

Fig. 1. Sketch of the background domain Ω_1 and foreground domain Ω_2 with the interface Σ.

disconnected subparts, thus generating two or more polyhedra with the foreground mesh in between. This situation occurs also at the corners of the overlapped region, where, independently of the thickness of the foreground mesh, it may happen that elements of the background mesh are split into disconnected polyhedra. On the other side, unlike the classical interface method of [1], our strategy allows us to use independent meshes, in particular to refine the foreground one without changing the background one.

The paper is organized as follows. In Section 2 we introduce the formulation for the Poisson equation on composite grids. In Section 3 we provide some details on the implementation of the proposed method in 3D. In Section 4 we show some 3D numerical tests to assess and validate the proposed method.

2. Numerical formulation

In this section, we introduce the numerical formulation of the problem we consider, i.e. the Poisson equation on composite meshes. Whereas at the continuous level the two resulting subdomains perfectly match the interface, we assume a different treatment of the corresponding meshes: the first one (*background mesh*) is fixed and in general does not match the interface; the second one (*overlapping mesh*) perfectly matches the interface.

2.1. Governing equations

Referring to Fig. 1, we consider two domains Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2 \subset \mathbb{R}^d$, $d = 2, 3$, and $\Sigma = \overline{\Omega}_1 \cap \overline{\Omega}_2$ is the common interface. In particular the foreground domain Ω_2 overlaps the background domain Ω_1. We denote by $\partial\Omega_1$ and $\partial\Omega_2$ the boundaries of the background and foreground domains, respectively, and we define $\Gamma_1 = \partial\Omega_1 \setminus \Sigma$ and $\Gamma_2 = \partial\Omega_2 \setminus \Sigma$. Finally, we indicate with $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ the outward unit normal to the domain Ω_1 and Ω_2, respectively. On the interface Σ we have $\boldsymbol{n}_1 = -\boldsymbol{n}_2 = \boldsymbol{n}$.

The continuous problem reads as follows: Given the functions $f_1 : \Omega_1 \to \mathbb{R}$ and $f_2 : \Omega_2 \to \mathbb{R}$, find the background solution $u_1 : \Omega_1 \to \mathbb{R}$ and the foreground solution $u_2 : \Omega_2 \to \mathbb{R}$, such that

$$-\nabla \cdot (\mu_1 \nabla u_1) = f_1 \quad \text{in } \Omega_1, \tag{1a}$$

$$u_1 = 0 \quad \text{on } \Gamma_1, \tag{1b}$$

$$-\nabla \cdot (\mu_2 \nabla u_2) = f_2 \quad \text{in } \Omega_2, \tag{1c}$$

$$u_2 = 0 \quad \text{on } \Gamma_2, \tag{1d}$$

$$u_1 = u_2 \quad \text{on } \Sigma, \tag{1e}$$

$$\mu_1 \nabla u_1 \cdot \boldsymbol{n} = \mu_2 \nabla u_2 \cdot \boldsymbol{n} \quad \text{on } \Sigma, \tag{1f}$$

where $T > 0$, μ_1 and μ_2 are the diffusion parameters, and where, for the sake of simplicity, we have considered homogeneous Dirichlet conditions on Γ_1 and Γ_2.

We consider the spaces $V_1 = H^1_{\Gamma_1}(\Omega_1) = \{v \in H^1(\Omega_1), v_{|\Gamma_1} = 0\}$ and $V_2 = H^1_{\Gamma_2}(\Omega_2) = \{v \in H^1(\Omega_2), v_{|\Gamma_2} = 0\}$. The weak formulation of the problem given by (1) reads as follows: find $(u_1, u_2) \in V_1 \times V_2$ such that $u_1 = u_2$ on Σ, and

$$a_1(u_1, v_1) + a_2(u_2, v_2) = (f_1, v_1)_{\Omega_1} + (f_2, v_2)_{\Omega_2},$$

Please cite this article in press as: L. Formaggia, et al., Unfitted extended finite elements for composite grids, Computers and Mathematics with Applications (2018), https://doi.org/10.1016/j.camwa.2018.05.028.