ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Mixed and discontinuous finite volume element schemes for the optimal control of immiscible flow in porous media[☆]

Sarvesh Kumar [a,*], Ricardo Ruiz-Baier [b], Ruchi Sandilya [c]

[a] Department of Mathematics, Indian Institute of Space Science and Technology, Thiruvananthapuram, 695 547, Kerala, India
[b] Mathematical Institute, University of Oxford, A. Wiles Building, Woodstock Road, Oxford OX2 6GG, United Kingdom
[c] Centre For Applicable Mathematics, Tata Institute of Fundamental Research, Bangalore 560065, India

ARTICLE INFO

Article history:
Received 23 October 2017
Received in revised form 24 May 2018
Accepted 27 May 2018
Available online xxxx

Keywords:
Optimal control problems
Immiscible displacement in porous media
Mixed formulations
Finite volume element methods
Error estimation

ABSTRACT

In this article we introduce a family of discretisations for the numerical approximation of optimal control problems governed by the equations of immiscible displacement in porous media. The proposed schemes are based on mixed and discontinuous finite volume element methods in combination with the optimise-then-discretise approach for the approximation of the optimal control problem, leading to nonsymmetric algebraic systems, and employing minimum regularity requirements. Estimates for the error (between a local reference solution of the infinite dimensional optimal control problem and its hybrid mixed/discontinuous approximation) measured in suitable norms are derived, showing optimal orders of convergence.

1. Introduction

Scope. We are interested in the accurate representation of the flow patterns produced by immiscible fluids within porous media. With the growing importance of the underlying physical processes in a variety of applications, the mathematical models used to describe this scenario have received a considerable attention in the past few decades. A popular example can be encountered in petroleum engineering, specifically in the standard process of oil recovery. The strategy there consists in injecting water (or other fluids having favourable density and viscosity properties) in such a way that the oil trapped in subsurface reservoirs is displaced mainly by pressure gradients. In its classical configuration, the technique of oil recovery by water injection employs two wells that contribute to maintain a high pressure and adequate flow rate in the oil field: an injection well from where the non-oleic liquid is injected, pushing the remaining oil towards a second, production well, from which oil is transported to the surface.

Regarding the simulation of these processes using mathematical models and numerical methods, there is a rich body of literature dealing with mixed finite element (FE) formulations where the filtration velocity and the pressure of each phase are solved at once (see, for instance, the classical works [1–4]). Mixed methods constructed using H(div)-conforming elements for the flux variable also allow for local mass conservation [5]. Alternative methods, also widely used in a variety of different formulations, include discontinuous Galerkin (DG) schemes which do not require inter-element continuity and feature element-wise conservation, arbitrary accuracy, controlled numerical diffusion, and can handle more adequately

☆ This work has been partially supported by the Engineering and Physical Sciences Research Council (EPSRC) through the research grant EP/R00207X/1 and by the London Mathematical Society through its Scheme 5, grant 51703.

* Corresponding author.
E-mail addresses: sarvesh@iist.ac.in (S. Kumar), ruizbaier@maths.ox.ac.uk (R. Ruiz-Baier), ruchi@tifrbng.res.in (R. Sandilya).

https://doi.org/10.1016/j.camwa.2018.05.031
0898-1221/

Please cite this article in press as: S. Kumar, et al., Mixed and discontinuous finite volume element schemes for the optimal control of immiscible flow in porous media, Computers and Mathematics with Applications (2018), https://doi.org/10.1016/j.camwa.2018.05.031.

problems with rough coefficients (see, for instance, [6] for a general overview on DG methods and [7–10] for their application in different configurations of multiphase flows).

A recurrent strategy in the design of numerical methods for coupled flow-transport problems as the one described above, is to combine different techniques with the objective of retaining the main properties of each compartmental scheme. For example, combined mixed FE and DG methods have been applied in [11,12,7] to numerically solve the coupled system of miscible displacement in porous media. On the other hand, a mixed finite volume element (FVE) method approximating the velocity–pressure pair and a discontinuous finite volume element (DFVE) scheme for the saturation equation are combined in [13]. FVE schemes require the definition of trial and test spaces associated with primal and dual partitions of the domain, respectively. Different types of dual meshes are employed when the FVE method is of conforming, non-nonconforming, or discontinuous type (see details and comparisons in e.g. [14–16]), but in most cases they feature local conservativity as well as suitability for deriving L^2-error estimates. We point out that schemes belonging to the particular class of DFVE approximations preserve features of both DG and general FVE methods, including smaller support of dual elements (when compared with conforming and non-conforming FVEs) and appropriateness in handling discontinuous coefficients.

Also in the context of FVE methods, the development in [17,18] uses a mixed (or hybrid) conforming–nonconforming discretisation applied to sedimentation problems, [19,20] analyse DFVE methods applied to viscous flow and degenerate parabolic equations, and [21] introduces mixed FE in combination with DFVE for a general class of multiphase problems. An extensive survey on different methods for multiphase multicomponent flows in porous media can be found in [22–24].

Optimal control and immiscible flow in porous media. Oil recovery in its so-called primary and secondary stages, can only lead to the extraction of 20%–40% of the reservoir's original oil. Other techniques (including a tertiary stage and the enhanced oil recovery process) can increase these numbers up to 30%–60%, but the development of control devices for manipulating the progression of the oil–water front, therefore increasing further the oil recovery, is still a topic of high interest. A viable approach consists in solving optimal control problems subject to the equations of two-phase incompressible immiscible flow in porous media. The goal is quite clear: to achieve optimal oil recovery from underground reservoirs after a fixed time interval. Several variables enter into consideration (as the price of oil and water, rock porosity and intrinsic permeability, the mobilities of the fluids, the constitutive relations defining capillary pressure, and so on) but here we will restrict the study to the adjustment of the water injection only.

Control theory and adjoint-based methods have been exploited in the optimisation of several aspects of the process, for instance in the design of valve operations for wells (see e.g. [25,26] and the review paper [27]). However, and in contrast with the situation observed for the approximation of direct systems, the numerical *analysis* of optimal control problems governed by incompressible flows in porous media (meaning rigorous error estimates and stability properties) has been so far restricted to classical discretisations. These include the FE method for immiscible displacement optimal control studied in [28] and the box method for the constrained optimal control problems with partially miscible two phase flow in porous media considered in [29]. Our goal here is to investigate optimal control problems governed by two-phase incompressible immiscible flow in porous media and their discretisation using a combined mixed FVE discretisation for the flow equations, and a DFVE scheme for the approximation of the transport equation. We concentrate our development on the optimise-then-discretise approach, where one first formulates the continuous optimality conditions and then the discretisation is applied to the continuous optimal system (see its applicability in similar scenarios in e.g. [30,31]).

Outline. The remainder of the paper is organised as follows. In Section 2 we state the model problem together with the corresponding optimality conditions, and present some preliminary results. This section also contains the main assumptions required on the model coefficients. Section 3 provides details about the discrete formulation, starting with the our mixed FVE/DFVE scheme applied to the optimal control problem under consideration. We also state useful properties of the discrete operators in Lemma 3.2, and finalise the section with the specification of the time discretisation scheme. In Section 4 we advocate the derivation of a priori error estimates in suitable norms. In fact, the main results of the paper are constituted by Theorems 4.3 and 4.4, where the *a priori* error estimates of optimal order are obtained for state, costate and control variables. Appendix A contains the proof of one auxiliary result needed for the error bounds, and Appendix B gives an overview of the implementation strategy employed in the solution of the overall optimal control problem.

2. Governing equations

We consider an optimal control problem governed by a nonlinear coupled system of equations representing the interaction of two incompressible fluids in a porous structure $\Omega \subset \mathbb{R}^2$. We study the process occurring within the time interval $J = (0, T]$, where the optimisation problem reads

$$\min_{q\in Q_{ad}} \mathcal{J}(q) := \frac{1}{2}\int_\Omega \tilde{w}c^2(T)\,\mathrm{d}\boldsymbol{x} + \frac{\alpha_0}{2}\int_0^T\int_\Omega \delta_0 q(t)^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t, \tag{2.1}$$

subject to

$$\begin{aligned} \boldsymbol{u} &= -\kappa(\boldsymbol{x})\lambda(c)\nabla p, && \forall(\boldsymbol{x},t)\in\Omega\times J,\\ \nabla\cdot\boldsymbol{u} &= (\delta_0-\delta_1)q(t), && \forall(\boldsymbol{x},t)\in\Omega\times J,\\ \phi\partial_t c - \nabla\cdot(\kappa(\boldsymbol{x})(\lambda\lambda_o\lambda_w p_c')(c)\nabla c) + \lambda_o'(c)\boldsymbol{u}\cdot\nabla c &= -\lambda_o(c)\delta_0 q(t), && \forall(\boldsymbol{x},t)\in\Omega\times J. \end{aligned} \tag{2.2}$$