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Numerical analysis of a leapfrog ADI–FDTD method for Maxwell's equations in lossy media

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ABSTRACT

Recently, a so-called one-step leapfrog ADI–FDTD method has been developed in engineering community for solving the 3D time-dependent Maxwell's equations. This method becomes quite popular in simulation wave propagation in graphene-based devices due to its efficiency. We investigate this method from a theoretical point of view by proving the energy conservation property, the unconditional stability of this ADI–FDTD method, and establishing the optimal second-order convergence rate in both time and space on non-uniform cubic grids. Numerical results are presented justifying our analysis.

1. Introduction

The alternating direction implicit (ADI) method was proposed by Douglas, Peaceman, Rachford back in 1955 [1,2], and it has been used very successfully for parabolic and elliptic PDEs [3,4]. An unconditionally stable and computationally efficient ADI method applying to 3D Maxwell's equations was introduced around 2000 by Zheng et al. [5,6] and Namiki [7]. This invention raised a lot of interest, and a large number of follow-up papers can be found in the literature, e.g. [8–11] and references therein. Recently, the so-called one-step leapfrog ADI–FDTD method has been developed [12–14]. It originated from the conventional ADI–FDTD but avoids mid-time computations. This leapfrog ADI–FDTD method becomes quite popular and finds many applications such as in simulating electromagnetic wave propagation in general dispersive media and graphene-based devices [15,16].

The influence of broad applications of ADI–FDTD methods has inspired some theoretical analysis of ADI–FDTD methods. The original proof of the unconditional stability of ADI method for Maxwell's equations given in [6] uses computational symbolic algebra to prove that all the eigenvalues of the 6×6 amplification matrix have magnitude one, which implies the unconditional stability. Later on, a simpler energy-based stability proof was given by Fornberg [8] for the ADI scheme on a periodic cube with arbitrary grid spacings in the three dimensions. Gao and Zhang [9] proved the optimal error estimates and energy conservation properties for the same ADI–FDTD method. Another rigorous convergence analysis of an ADI splitting method for Maxwell's equations is recently proved in the framework of operator semigroup theory by Hochbruck et al. [10].

As for the recently proposed leapfrog ADI–FDTD method, Gao and Zheng [14] use the von Neumann method combining the Jury criterion and the computational symbolic software Mathematica to show that the amplification factor is less or equal to unity in magnitude, which leads to the unconditional stability of the scheme. Other than this theoretical result, we are unaware of any other theoretical analysis of this popular leapfrog ADI–FDTD scheme. Here we try to fill the gap. More

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specifically, in this paper we use the classic energy method to prove the unconditional stability of this leapfrog ADI–FDTD method, and establish the optimal second-order convergence rate in both time and space on non-uniform cubic grids. We remark that the analysis is nontrivial, since we have to sort out so many terms to come up with the right discrete energy norm.

The rest part of the paper is organized as follows. In Section 2, we derive leapfrog ADI–FDTD method for Maxwell's equations in lossy media. In Section 3, we first prove an energy conservation law for this ADI–FDTD scheme with periodic boundary conditions, which immediately yields the unconditional stability of the scheme. Then we prove the second-order error estimate under some discrete energy norms for the ADI–FDTD scheme. In Section 4, we present some numerical examples to confirm our theoretical results. We conclude the paper in Section 5.

2. The leapfrog ADI–FDTD method

To make our paper easy to follow and self-contained, we first present the construction of the leapfrog ADI–FDTD scheme by following the original papers [12–14]. Consider the time-dependent Maxwell's equations in a linear, lossy, and non-dispersive medium with permittivity $\boldsymbol{\epsilon}$, permeability $\boldsymbol{\mu}$, and electric conductivity $\boldsymbol{\sigma}$ written as follows:

$$\boldsymbol{\epsilon}\partial_t \boldsymbol{E} = (A-B)\boldsymbol{H} - \boldsymbol{\sigma}\boldsymbol{E}, \quad \Omega \times (0,T), \tag{1}$$

$$\boldsymbol{\mu}\partial_t \boldsymbol{H} = (B-A)\boldsymbol{E}, \quad \Omega \times (0,T), \tag{2}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ denote the electric and magnetic fields, respectively, $\partial_t u$ for the partial derivative of u with respect to variable t, and

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \mu_x & 0 & 0 \\ 0 & \mu_y & 0 \\ 0 & 0 & \mu_z \end{bmatrix}, \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 0 & \partial_y \\ \partial_z & 0 & 0 \\ 0 & \partial_x & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \partial_z & 0 \\ 0 & 0 & \partial_x \\ \partial_y & 0 & 0 \end{bmatrix}.$$

For simplicity, we assume that domain $\Omega = [a,b] \times [b,c] \times [e,f]$, and time interval $[0,T]$. Furthermore, we divide the domain Ω by a nonuniform cubic grid :

$$a = x_0 < x_1 < \cdots < x_{N_x} = b, \ c = y_0 < y_1 < \cdots < y_{N_y} = d, \ e = z_0 < z_1 < \cdots < z_{N_z} = f,$$

and divide the time interval $[0,T]$ into N_t uniform intervals, i.e., we have discrete time $t_n = n\tau, \tau = \frac{T}{N_t}, n = 0, 1, \ldots, N_t$, grid points $x_i = ih_x, h_x = \frac{b-a}{N_x}, i = 0, 1, \ldots, N_x$ in the x-direction, grid points $y_j = jh_y, h_y = \frac{d-c}{N_y}, j = 0, 1, \ldots, N_y$ in the y-direction, and grid points $z_k = kh_z, h_z = \frac{f-e}{N_z}, k = 0, 1, \ldots, N_z$ in the z-direction. Note that h_x, h_y and h_z can be different.

To develop the so-called one-step leapfrog ADI–FDTD method for (1)–(2), we first consider an ADI–FDTD method in two steps:

Step 1:

$$\boldsymbol{\epsilon}\boldsymbol{E}^{n+\frac{1}{2}} = \boldsymbol{\epsilon}\boldsymbol{E}^n + \frac{\tau}{2}(A\boldsymbol{H}^{n+\frac{1}{2}} - B\boldsymbol{H}^n - \boldsymbol{\sigma}\boldsymbol{E}^{n+\frac{1}{2}}), \tag{3}$$

$$\boldsymbol{\mu}\boldsymbol{H}^{n+\frac{1}{2}} = \boldsymbol{\mu}\boldsymbol{H}^n + \frac{\tau}{2}(B\boldsymbol{E}^{n+\frac{1}{2}} - A\boldsymbol{E}^n). \tag{4}$$

Step 2:

$$\boldsymbol{\epsilon}\boldsymbol{E}^{n+1} = \boldsymbol{\epsilon}\boldsymbol{E}^{n+\frac{1}{2}} + \frac{\tau}{2}(A\boldsymbol{H}^{n+\frac{1}{2}} - B\boldsymbol{H}^{n+1} - \boldsymbol{\sigma}\boldsymbol{E}^{n+\frac{1}{2}}), \tag{5}$$

$$\boldsymbol{\mu}\boldsymbol{H}^{n+1} = \boldsymbol{\mu}\boldsymbol{H}^{n+\frac{1}{2}} + \frac{\tau}{2}(B\boldsymbol{E}^{n+\frac{1}{2}} - A\boldsymbol{E}^{n+1}). \tag{6}$$

Substituting $\boldsymbol{H}^{n+\frac{1}{2}}$ of (4) into (3), we have

$$\begin{aligned} \boldsymbol{\epsilon}\boldsymbol{E}^{n+\frac{1}{2}} &= \boldsymbol{\epsilon}\boldsymbol{E}^n + \frac{\tau}{2}A\left[\boldsymbol{H}^n + \frac{\tau}{2}\boldsymbol{\mu}^{-1}(B\boldsymbol{E}^{n+\frac{1}{2}} - A\boldsymbol{E}^n)\right] - \frac{\tau}{2}(B\boldsymbol{H}^n + \boldsymbol{\sigma}\boldsymbol{E}^{n+\frac{1}{2}}) \\ &= \boldsymbol{\epsilon}\boldsymbol{E}^n + \frac{\tau}{2}A\boldsymbol{H}^n + \frac{\tau^2}{4}A\boldsymbol{\mu}^{-1}B\boldsymbol{E}^{n+\frac{1}{2}} - \frac{\tau^2}{4}A\boldsymbol{\mu}^{-1}A\boldsymbol{E}^n - \frac{\tau}{2}B\boldsymbol{H}^n - \frac{\tau}{2}\boldsymbol{\sigma}\boldsymbol{E}^{n+\frac{1}{2}}. \end{aligned} \tag{7}$$

Substituting $\boldsymbol{H}^{n+\frac{1}{2}}$ of (6) into (5) and reducing all n's by 1, we have

$$\begin{aligned} \boldsymbol{\epsilon}\boldsymbol{E}^{n} &= \boldsymbol{\epsilon}\boldsymbol{E}^{n-\frac{1}{2}} + \frac{\tau}{2}A\left[\boldsymbol{H}^n - \frac{\tau}{2}\boldsymbol{\mu}^{-1}(B\boldsymbol{E}^{n-\frac{1}{2}} - A\boldsymbol{E}^n)\right] - \frac{\tau}{2}(B\boldsymbol{H}^n + \boldsymbol{\sigma}\boldsymbol{E}^{n-\frac{1}{2}}) \\ &= \boldsymbol{\epsilon}\boldsymbol{E}^{n-\frac{1}{2}} + \frac{\tau}{2}A\boldsymbol{H}^n - \frac{\tau^2}{4}A\boldsymbol{\mu}^{-1}B\boldsymbol{E}^{n-\frac{1}{2}} + \frac{\tau^2}{4}A\boldsymbol{\mu}^{-1}A\boldsymbol{E}^n - \frac{\tau}{2}B\boldsymbol{H}^n - \frac{\tau}{2}\boldsymbol{\sigma}\boldsymbol{E}^{n-\frac{1}{2}}. \end{aligned} \tag{8}$$