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The lumped mass finite element method for surface parabolic problems: Error estimates and maximum principle☆

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ABSTRACT

The lumped mass method is extended to the surface finite element method for solving the surface parabolic equations. The main purpose of the proposed method is to overcome the difficulty that the surface finite element method does not guarantee the maximal principle of the surface heat equation. Optimal error estimates are given for the semi-discrete and fully-discrete schemes of the proposed method respectively. The maximum principle is shown for surface heat equations and its preservation by the lumped mass surface finite element under the Delaunay type triangulation. Moreover, some results of positivity and monotonicity are derived for nonlinear parabolic equations. Finally some numerical experiments are displayed to illustrate the validity and numerical performance of the proposed method.

1. Introduction

1.1. Motivation

Nowadays, the partial differential equation models on surfaces are constructed for many applications in material science, mathematical biology and the mathematics of images where the surface parabolic equations are often applied to describe the heat transfer and phase-transfer on ultra-thin materials and solid surface [1–6]. In real application, to solve surface partial differential equations analytically is more difficult than the work on general 2D/3D domains since the domains of the equations are curved. Hence, the surface finite element method (FEM) [2,3,7] was proposed to solve the surface equations numerically.

Comparing the surface FEM with the standard FEM for parabolic equations, they have the same shortage in preserving the maximum principle of the homogeneous parabolic equations (heat equation), see [8] and examples in Section 3.2. The lack of a discrete maximum principle in the standard method is mainly caused by the nondiagonal mass matrix. For the standard FEM, a modified method called the lumped mass FEM was provided and analysed in terms of the error estimates [8–14] and the maximum principle preservation [8,11,12,15]. The main idea of lumped mass method is to modify the mass matrix of standard FEM to a diagonal matrix whose diagonal elements are the row sums of original mass matrix. This method was

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shown to be second-order in space, and the maximum principle can be guaranteed for the heat equation. In this paper, we extend the lumped mass method to the surface parabolic equations, and show the error estimates and maximum principle preservation.

The layout of the paper is as follows. In the rest of Section 1, we introduce some basic concepts of surface parabolic equations and the surface FEM. In Section 2, the lumped mass FEM is proposed for surface parabolic equations and the related error estimates for semi-discrete, backward Euler and Crank–Nicolson schemes are shown respectively. In Section 3, firstly, the maximum principle is shown for the homogeneous parabolic equations on bounded surfaces. Then some examples are given to illustrate that the maximum principle cannot be guaranteed by the surface FEM. Moreover, under the Delaunay type triangulation, the maximum principle guaranteed by the lumped mass FEM is shown. At the end of the section, some results of maximum principle, positivity and monotonicity for nonlinear parabolic equations are derived. In Section 4, some numerical experiments are given to illustrate numerical performance of the proposed method including tests of convergence and maximum principle. Finally, some conclusions are drawn.

1.2. Finite element method for surface parabolic equations

Let Γ be a compact, connected and oriented hypersurface contained in bounded domain $\Omega \subset \mathbb{R}^3$ with $\partial\Gamma$ empty. Suppose that there exists a function $\phi \in C^2(\Omega)$ such that Γ can be expressed as a zero-level set function form, that is,

$$\Gamma = \{\mathbf{x} \in \Omega \mid \phi(\mathbf{x}) = 0\}.$$

The normal vector field of Γ is defined as

$$\mathbf{n}(x) = \frac{\nabla\phi(\mathbf{x})}{|\nabla\phi(\mathbf{x})|} = (n_1, n_2, n_3)^{\mathbf{T}},$$

where function ϕ is required to satisfy $\nabla\phi(\mathbf{x}) \neq 0$, ∇ denotes the standard gradient operator in $\mathbb{R}^3$.

Now we consider the initial value problem

$$\begin{cases} u_t - \Delta_\Gamma u = f, & \mathbf{x} \in \Gamma,\ t \in (0, T], \\ u|_{t=0} = u_0, & \mathbf{x} \in \Gamma, \end{cases} \tag{1.1}$$

where Δ_Γ is the Laplace–Beltrami operator defined by $\nabla_\Gamma \cdot \nabla_\Gamma$ and ∇_Γ denotes the surface or tangential gradient such that

$$\nabla_\Gamma \eta(\mathbf{x}) = \nabla\eta(\mathbf{x}) - \nabla\eta(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\mathbf{n}(\mathbf{x}) = P(\mathbf{x})\nabla\eta(\mathbf{x}).$$

The matrix P satisfies $P_{ij}(\mathbf{x}) = \delta_{ij} - n_i(\mathbf{x})n_j(\mathbf{x})$ with the property $P\nabla_\Gamma = \nabla_\Gamma$.

For solving (1.1), the first step is to choose the vertices $\{X_j\}_{j=1}^{N_p}$ on Γ to generate a piecewise triangulated surface element Γ_h consisting of non-overlapped and quasi-uniform triangles e_h. Then the original surface is replaced by this triangulated surface. From this step, we can get a triangulated mesh $\mathcal{T}_h = \{e_h\}$ with size h which is the maximum diameter of each triangle. Then, a piecewise bijection between point $a \in \Gamma$ and $\mathbf{x} \in \Gamma_h$ can be established by $\mathbf{x} = a(\mathbf{x}) + \bar{d}(\mathbf{x})\mathbf{n}(a(\mathbf{x}))$ with sufficient small h, where $\bar{d}(\mathbf{x})$ is a signed distance function satisfying $|\bar{d}(\mathbf{x})| = \mathrm{dist}(\mathbf{x}, \Gamma)$, $\nabla\bar{d}(\mathbf{x}) = \mathbf{n}(a(\mathbf{x}))$ and $|\nabla\bar{d}(\mathbf{x})| = 1$. Hence, for a function η defined on Γ_h, we can lift it onto Γ by

$$\eta^l(a) = \eta(\mathbf{x}(a)), \qquad a \in \Gamma.$$

We can understand by $\eta^l(\mathbf{x})$ the constant extension from Γ in the normal direction $\mathbf{n}(a(\mathbf{x}))$ and we extend the normal constantly such that $\mathbf{n}(\mathbf{x}) = \mathbf{n}(a(\mathbf{x}))$. We can also define the inverse lift such that $\zeta^{-l}(\mathbf{x}) = \zeta(a(\mathbf{x}))$ for a function ζ defined on Γ.

Set $p \in [1, \infty)$ and let $L^p(\Gamma)$ denote the space of functions $\eta : \Gamma \to \mathbb{R}$ satisfying

$$\|\eta\|_{L^p(\Gamma)} = \left(\int_\Gamma |\eta|^p dA\right)^{\frac{1}{p}} < \infty.$$

Let $W^{m,p}(\Gamma)$ denote the space of functions $\eta : \Gamma \to \mathbb{R}$ with weak tangential derivatives up to order m in $L^p(\Gamma)$. Specially, for $p = 2$, we write $H^m(\Gamma)$ for $W^{m,2}(\Gamma)$. For any functions $w, v \in H^1(\Gamma)$, we denote the following inner product

$$(w, v)_\Gamma = \int_\Gamma wv\, dA \quad \text{and} \quad (\nabla_\Gamma w, \nabla_\Gamma v)_\Gamma = \int_\Gamma \nabla_\Gamma w \cdot \nabla_\Gamma v\, dA.$$

Similarly, we can define norm, inner product, and other quantities in the discrete case such that the function spaces $L^2(\Gamma_h)$ and $H^1(\Gamma_h)$ with the inner products $(W, V)_{\Gamma_h}$ and $(\nabla_{\Gamma_h} W, \nabla_{\Gamma_h} V)_{\Gamma_h}$ for any functions $W, V \in H^1(\Gamma_h)$ respectively.

In the numerical analysis framework of the surface finite element method, the following notations and lemmas are important to establish the stability and error analyses. Let n_h be the normal vector of Γ_h, which is constant on each triangle e_h. Using a chain rule for differentiation, we have

$$\begin{aligned} \nabla_{\Gamma_h}\eta(\mathbf{x}) &= P_h(\mathbf{x})\nabla\eta^l(a(\mathbf{x})) = P_h(\mathbf{x})\left(\mathcal{I} - \bar{d}(\mathbf{x})\mathcal{H}(\mathbf{x}) - \mathbf{n}^{\mathbf{T}}(a(\mathbf{x}))\mathbf{n}(a(\mathbf{x}))\right)\nabla\eta^l(a(\mathbf{x})) \\ &= P_h(\mathbf{x})(\mathcal{I} - \bar{d}(\mathbf{x})\mathcal{H}(\mathbf{x}) - \mathbf{n}^{\mathbf{T}}(\mathbf{x})\mathbf{n}(\mathbf{x}) + \bar{d}(\mathbf{x})(\mathcal{H}(\mathbf{x})\mathbf{n}(\mathbf{x}))^{\mathbf{T}}\mathbf{n}(\mathbf{x}))\nabla\eta^l(a(\mathbf{x})) \end{aligned}$$

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