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Bounds for blow-up time of a reaction–diffusion equation with weighted gradient nonlinearity

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ABSTRACT

In this paper, we focus on the bounds for blow-up time of null Dirichlet initial boundary value problem for a reaction–diffusion equation with weighted gradient nonlinearity. By virtue of the method of super-sub solution and the technique of modified differential inequality, we establish sufficient conditions to guarantee that the solution blows up at finite time under appropriate measure sense. Meanwhile, upper and lower bounds for the blow-up time are found in higher dimensional spaces and some examples for application are presented.

1. Introduction

Our main interest lies in the following reaction–diffusion equation with weighted gradient source terms

$$u_t = \Delta u + a(x)f(|\nabla u|), \quad (x,t) \in \Omega \times (0,t^*), \tag{1.1}$$

subject to the homogeneous Dirichlet boundary and initial conditions

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,t^*), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$ and t^* represents the blow-up time when blow-up occurs, otherwise $t^* = +\infty$. The nonlinearity $f(|\nabla u|)$ is assumed to be a nonnegative continuous function which satisfies some appropriate conditions and the weight function $a \in C^0(\bar{\Omega})$ is also assumed to satisfy

(a_1) $a(x) > 0, x \in \Omega$ and $a(x) = 0, x \in \partial\Omega$,

or

(a_2) $a(x) \geq c > 0$ for all $x \in \bar{\Omega}$.

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Moreover, the nonnegative initial data $u_0(x)$ is a C^1-function which satisfies a compatibility condition. Therefore, by the standard parabolic theory, one can deduce that problem (1.1)–(1.3) admits a unique nonnegative classical solution $u(x,t)$, whose maximal existence time $t^* \in (0, +\infty]$. In addition, if $t^* < +\infty$, then $u(x,t)$ blows up at finite time in C^1-norm (cf. [1, Theorem 10, p. 206]); that is,

$$\limsup_{t \to t^*} \sup_{x \in \bar{\Omega}} \{|u(x,t)| + |\nabla u(x,t)|\} = +\infty.$$

Model (1.1) is often referred to as a viscous Hamilton–Jacobi equation. Also, (1.1) is related with the Kardar–Parisi–Zhang equation in the physical theory of growth and roughening of surfaces, see [2,3] and references therein for details. Note that it is one of the simplest examples of parabolic equation with a nonlinearity depending on the first-order spatial derivatives, and it can be considered as an analogue of the extensively studied equation with zero-order nonlinearity, $u_t = \Delta u + u|u|^{p-1},\ p > 1$, for which one can refer to monographs as well as the survey paper [4–8].

Particularly, Quittner and Souplet [6, Chapters 3, 4] introduced the qualitative properties of the solution to the reaction–diffusion equation with homogeneous Dirichlet boundary condition and constant coefficients, i.e., the weight function $a(x) = a > 0$, in detail. Roughly, the occurrence and type of blow-up depend on the constant a, the initial data, and the domain. Furthermore, another notable feature of gradient model is that gradient blow-up occurs on the boundary or interior under suitable conditions.

In this article, we would like to investigate the blow-up phenomena for a gradient model with variable coefficient, focusing on deriving the blow-up conditions and the bounds for blow-up time of an initial and homogeneous Dirichlet boundary value problem. At present, there are many literatures and methods on bounds for blow-up time of the reaction–diffusion model with constant coefficients or time-dependent coefficients and zero-order nonlinearity, or with a competitive relationship between zero-order and first-order nonlinearities. In addition, the lower bounds for the blow-up time are mostly derived in the three-dimensional space and the main difficulty lies in determination of Sobolev optimal constant. One can refer to [9–13] (without gradient term), [14–17] (with gradient term), and references therein. Specially, Hesaaraki and Moameni [14, Sec. 2] pointed out that the gradient model with a constant coefficient $u_t = \Delta u + |\nabla u|^p$ can give the similar blow-up phenomena as the one with zero-order nonlinearity, except gradient blow-up.

Concerning the research on the reaction–diffusion model with spatial variable coefficients, one can refer to [18–20] and the references therein. Ma and Fang [18] studied the following semilinear reaction–diffusion equation with weighted inner source terms

$$u_t = \Delta u + a(x)f(u), \quad (x,t) \in \Omega \times (0, t^*), \tag{1.4}$$

under Robin boundary condition, where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with smooth boundary, the weight function $a(x)$ satisfies (a_1) or (a_2), and the nonlinearity $f(u)$ satisfies appropriate nonlocal conditions. Under some suitable conditions, they established sufficient conditions for solutions of global existence and blowing up at finite time, and derived bounds for the blow-up time of the solution in higher dimensional spaces under appropriate measure sense. Meanwhile, they [19,20] considered the reaction model with local or nonlocal inner absorption and nonlinear boundary flux.

Briefly, much less effort has been devoted to blow-up analysis for the reaction–diffusion model (1.1)–(1.3) with weighted gradient source term in the higher dimensional spaces ($N \geq 1$), to our knowledge. At a glance, the main difficulty lies in finding the influence of weight function $a(x)$ and gradient source term to the blow-up phenomena. In this article, our main goal is to establish sufficient conditions to guarantee occurrence of blow-up at finite time for problem (1.1)–(1.3) under appropriate measure sense. Moreover, upper and lower bounds for the blow-up time are derived in higher dimensional spaces.

The rest of our paper is organized as follows: In Section 2, we assume some appropriate conditions on the weight function $a(x)$ and nonlinearity $f(|\nabla u|)$ to derive some sufficient conditions to ensure the solution blows up at finite time for problem (1.1)–(1.3) by constructing a suitable sub-solution and using the technique of modified differential inequality, and obtain upper bounds for the blow-up time. In Section 3, we are devoted to drive a lower bound for the blow-up time by constructing a suitable blow-up super-solution. Finally, a few examples are given to illustrate applications of our main results in Section 4.

2. Blow-up and upper bounds for t^*

In fact, similar to the qualitative properties of reaction–diffusion model with zero-order nonlinearity, we can establish the super-solution such as $\bar{u}(x,t) = e^{kt}$, where k is a positive constant. It is not difficult to verify that the null Dirichlet initial boundary value problem with a weighted first-order nonlinearity, $u_t = \Delta u + a(x)|\nabla u|^p$ when $p > 2$, has a global solution for sufficiently small initial data. Hence, in this section, we derive the sufficient conditions to ensure that the solution of problem (1.1)–(1.3) with appropriately large initial data blows up at finite time and obtain upper bounds for the blow-up time under some different measure. Meanwhile, adding a positive constant to the right-hand side of (1.1), we can derive some blow-up result for any nonnegative and not identically zero initial data.

First of all, we derive some blow-up properties in L^2-norm.

Theorem 2.1. *Let $u(x,t)$ be a nonnegative classical solution of problem (1.1)–(1.3). Suppose that the nonnegative integrable function f satisfies the following condition:*

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