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A uniformly and optimally accurate multiscale time integrator method for the Klein–Gordon–Zakharov system in the subsonic limit regime

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ABSTRACT

We present a uniformly and optimally accurate numerical method for discretizing the Klein–Gordon–Zakharov system (KGZ) with a dimensionless parameter $0 < \varepsilon \leq 1$, which is inversely proportional to the acoustic speed. In the subsonic limit regime, i.e., $0 < \varepsilon \ll 1$, the solution of KGZ system propagates waves with $O(\varepsilon)$- and $O(1)$-wavelength in time and space, respectively, and rapid outspreading initial layers with speed $O(1/\varepsilon)$ in space due to the singular perturbation of the wave operator in KGZ and/or the incompatibility of the initial data. Based on a multiscale decomposition by frequency and amplitude, we propose a multiscale time integrator Fourier pseudospectral method by applying the Fourier spectral discretization for spatial derivatives followed by using the exponential wave integrator in phase space for integrating the decomposed system at each time step. The method is explicit and easy to be implemented. Extensive numerical results show that the MTI-FP method converges optimally in both space and time, with exponential and quadratic convergence rate, respectively, which is uniformly for $\varepsilon \in (0, 1]$. Finally, the method is applied to study the convergence rates of the KGZ system to its limiting models in the subsonic limit and wave dynamics and interactions of the KGZ system in 2D.

1. Introduction

Consider the coupled Klein–Gordon–Zakharov (KGZ) equations in nondimensional variables:

$$\begin{aligned}&\gamma^2\partial_{tt}\psi(\mathbf{x},t)-\Delta\psi(\mathbf{x},t)+\frac{1}{\gamma^2}\psi(\mathbf{x},t)+\phi(\mathbf{x},t)\psi(\mathbf{x},t)=0,\\&\varepsilon^2\partial_{tt}\phi(\mathbf{x},t)-\Delta\phi(\mathbf{x},t)-\Delta|\psi|^2(\mathbf{x},t)=0,\quad \mathbf{x}\in\mathbb{R}^d,\quad t>0,\end{aligned}\tag{1.1}$$

where t is time, $\mathbf{x} \in \mathbb{R}^d$ $(d = 1, 2, 3)$ is the spatial coordinate, $\psi := \psi(\mathbf{x}, t)$ is a complex-valued function representing the fast timescale component of the electric field raised by electrons and $\phi := \phi(\mathbf{x}, t)$ is a real function of the deviation of the ion density from its equilibrium; $\gamma, \varepsilon \in (0, 1]$ are two dimensionless parameters which are inversely proportional to the plasma frequency and speed of sound, respectively. It takes an important role in the investigation of the dynamics of strong Langmuir turbulence in plasma physics [1,2]. Moreover, it can be derived from the two-fluid Euler–Maxwell equations (cf. [1,3–5] for physical and formal derivations, and [6] for mathematical justifications).

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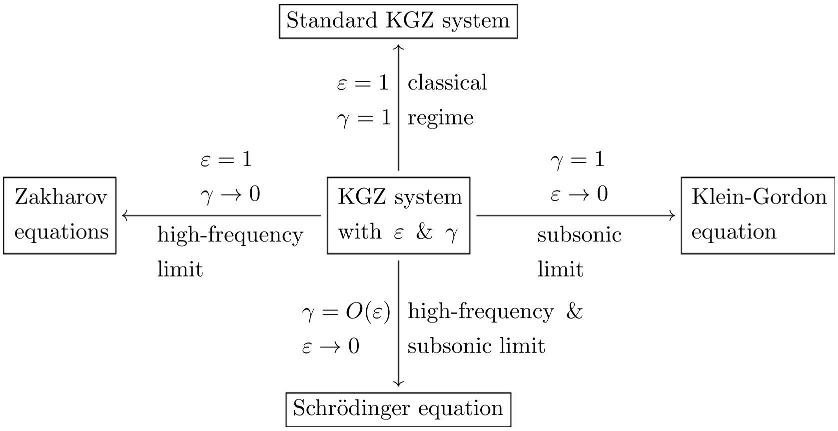

Fig. 1.1. Diagram of different limits of the KGZ system (1.1).

As it is known, (1.1) is time symmetric or time reversible and conserves the total *energy* (or *Hamiltonian*) [2,7]

$$\mathcal{H}(t) := \int_{\mathbb{R}^d} \left[\gamma^2|\partial_t\psi|^2 + |\nabla\psi|^2 + \frac{1}{\gamma^2}|\psi|^2 + \frac{\varepsilon^2}{2}|\nabla\varphi|^2 + \frac{1}{2}|\phi|^2 + \phi|\psi|^2\right]d\mathbf{x} \equiv \mathcal{H}(0), \quad t \geq 0, \tag{1.2}$$

where $\varphi := \varphi(\mathbf{x}, t)$ is defined by $\Delta\varphi(\mathbf{x}, t) = \partial_t\phi(\mathbf{x}, t)$ with $\lim_{|\mathbf{x}|\to\infty}\varphi(\mathbf{x}, t) = 0$.

Different parameter regimes could be considered for the KGZ system (1.1) which is displayed in Fig. 1.1:

- Standard regime, i.e., $\gamma = 1$ and $\varepsilon = 1$, there were extensive analytical and numerical studies for the KGZ equations (1.1) with $\varepsilon = \gamma = 1$ in the last two decades. For the derivation of the KGZ system from two-fluid Euler–Maxwell system, we refer to [3,6]; and for the well-posedness of the Cauchy problem, we refer to [8–11]; for the stability behavior of the KGZ system, we refer to [10,12–14]. For the numerical part, we refer to [15,16] for finite difference method and [17,18] for exponential-wave-integrator Fourier pseudospectral method.
- High-plasma-frequency limit regime, i.e., $\varepsilon = 1$ and $0 < \gamma \ll 1$, the KGZ system (1.1) reduces to the vectorial Zakharov system [2,7]

$$\begin{cases} 2i\partial_t u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) + \phi(\mathbf{x}, t)u(\mathbf{x}, t) = 0, \\ \varepsilon^2\partial_{tt}\phi(\mathbf{x}, t) - \Delta\phi(\mathbf{x}, t) - \Delta|\psi(\mathbf{x}, t)|^2 = 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \end{cases} \tag{1.3}$$

where $u = (u_1, u_2) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{C} \times \mathbb{C}$. In fact, by plugging $\psi(\mathbf{x}, t) = e^{it/\gamma^2}u_1 + e^{-it/\gamma^2}\overline{u_2}$, we can eliminate the diverging term $\frac{1}{\gamma^2}\psi$ in (1.1) to obtain

$$\gamma^2\partial_{tt}u(\mathbf{x}, t) + 2i\partial_t u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) + \phi(\mathbf{x}, t)u(\mathbf{x}, t) = 0.$$

Taking the limit $\gamma \to 0$, one can get the Zakharov system (1.3). In the high-plasma-frequency limit regime, the solution propagates waves with amplitude at $O(1)$ and wavelength at $O(\varepsilon^2)$ in time and $O(1)$ in space. A multiscale time integrator sine pseudospectral method for the KGZ system was proposed in [18] and extensive numerical studies suggest that the method is uniformly and optimally accurate in space with exponential convergence rate and uniformly in time with linear convergence rate. Very recently, the authors [19] presented a novel class of oscillatory integrators for the KGZ system, which are uniformly accurate in the high-plasma-frequency limit regime and establish the error estimates rigorously.
- Subsonic limit regime, i.e., $\gamma = 1$ and $0 < \varepsilon \ll 1$, the KGZ system (1.1) converges to the Klein–Gordon equation [20]

$$\partial_{tt}\psi(\mathbf{x}, t) - \Delta\psi(\mathbf{x}, t) + \psi(\mathbf{x}, t) - |\psi(\mathbf{x}, t)|^2\psi(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0. \tag{1.4}$$

When $0 < \varepsilon \ll 1$, the solution is oscillatory in time at $O(\varepsilon)$-wavelength. It was proved in [21] that the classical conservative finite difference method requires the ε-scalability is mesh size $h = O(\varepsilon^{1/2})$ and time step $\tau = O(\varepsilon^{3/2})$. Based on an asymptotic consistent formulation, a uniformly accurate finite difference method was proposed [22] and it was analyzed that the method converges in space and time with quadratic and linear convergence rates, respectively, which are uniformly for $0 < \varepsilon \leq 1$.
- Simultaneously high-plasma-frequency and subsonic limit regimes, i.e., $\gamma \lesssim \varepsilon$ and $0 < \varepsilon \ll 1$, the KGZ system (1.1) converges to the vectorial nonlinear Schrödinger equation [2]:

$$2i\partial_t u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) - |u(\mathbf{x}, t)|^2u(\mathbf{x}, t) = 0, \quad u = (u_1, u_2) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{C} \times \mathbb{C}. \tag{1.5}$$

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