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An adaptive mesh refinement strategy for finite element solution of the elliptic problem

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ABSTRACT

In this paper we develop an adaptive finite element method for elliptic problems. First, we assume that in each subdomain the norm of the approximation error at the current mesh configuration is bounded by the norm of the approximation error obtained at the previous mesh configuration, for some norm H^s. Then an *a-posteriori* error estimator is designed based on the approximate solution correction between the solution on the last two mesh configurations. Based on this new error estimator, the element-wise refinement strategy in each subdomain is provided for a given tolerance.

A discussion on the choice of the coefficients in the assumption is given for different norm spaces and for different degrees of finite element family. Four 2D numerical benchmark examples of different domains and two 3D numerical benchmark examples are tested to demonstrate the robustness of our method. When possible, our numerical results are also compared to corresponding results from existing methods. All the results show that the proposed method is robust and efficient in terms of the number of degrees of freedom.

1. Introduction

The adaptive finite element method for solving elliptic problems has been studied for many years due to the advantage of a significant reduction of the computational cost within a pre-established tolerance [1–3]. In addition, this method can be easily incorporated with domain decomposition techniques to improve the efficiency of the solver, see for example [4–7]. The development of adaptive finite element method relies on two important techniques, namely the adaptive mesh refinement (AMR) strategy and the error estimator. The AMR strategy can be broadly classified into three categories: *h*-type refinement, *p*-type refinement and *r*-type refinement. The *h*-type refinement is to refine the meshes in the region where the error is relatively large whereas coarsen the meshes in the region where the error is relatively small [8,9]. The *p*-type refinement, instead, increases the order of the polynomial functions instead of directly refining the mesh [10,11]. The *r*-type refinement is to move the nodes to increase the mesh density in the region of interest without changing the number of nodes or cells present in a mesh or changing the connectivity of a mesh [12–14]. Some hybrid refinement strategies are also developed such as *hp*-type and *hr*-type refinements, see [15–17].

The above AMR strategies are performed according to the local error indicator, called error estimator, which decides whether the local mesh should be refined or not. There exist two types of error estimators: the *a-priori* error estimator and *a-posteriori* error estimator. The *a-priori* error estimator provides the error estimate by using well-chosen approximation of the exact solution but not an actual error estimate for a given mesh. It is not so convenient to apply this method to complicated problems for which little information about exact solutions is provided resulting in being hard to seek

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appropriate approximation of the exact solutions. In contrast, the *a-posteriori* error estimator uses the approximate solution itself to construct the error estimate which can be directly computed based on the approximate solution on a given mesh. Thus due to its reliability and ease of implementation, after the pioneering work of Babuška and Rheinboldt [18], theorems and methods of the *a-posteriori* error estimator have been developed extensively, see [19–23]. The *a-posteriori* error estimator can also be classified into three classes, namely explicit error estimators, implicit error estimators, and recovery based error estimators, see [20] and the references therein.

In this work, we use the h-type and local midpoint as refinement strategy. We develop a new *a-posteriori* error estimator as local error indicator for the refinement of the mesh. The estimator is based on the assumption that in each subdomain the norm of the new approximation error at the current mesh configuration is bounded by the norm of the approximation error at the previous mesh configuration, for some norm H^s. It may be the case that at a given level and for some elements, the approximation error is much smaller than the required given tolerance. In this case, an over-conservative mesh configuration is obtained. Instead of coarsening the regions that have been overly refined, we prefer to ease the tolerances in the regions that have to be refined again, so that the global tolerance remains the same. A series of benchmark tests are used to validate the robustness and efficiency of the algorithm. The new adaptive finite element method is implemented in FEMuS [24], an open-source finite element C++ library built on top of PETSc [25].

The outline of this paper is organized as follows. First, we introduce the domain discretization and define some notations. Then, the new *a-posteriori* error estimator is presented. Based on the error estimator, two new theorems are proven. These give an effective way to define the element-wise refinement strategy in each subdomain for any given tolerance. Then, multiple numerical benchmark examples are tested and their results are discussed. Finally, some conclusions are made.

2. Error estimator for elliptic problems

We consider the Poisson problem as the model problem. This is an example of elliptic problem. The Poisson equation is given

$$-\Delta u = f, \quad \text{in } \Omega, \tag{1}$$

$$u = g, \quad \text{on } \partial\Omega, \tag{2}$$

on a bounded domain $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$, where f is the source term, and g is some given boundary condition. For nonzero Dirichlet boundary conditions, we can easily transform the problem to the case of homogeneous Dirichlet boundary conditions defining $\widetilde{u} = u - h$, where h is any smooth function, whose trace on the boundary is g, then

$$-\Delta\widetilde{u} = f + \Delta h = \widetilde{f}, \quad \text{in } \Omega,$$

$$\widetilde{u} = 0, \quad \text{on } \partial\Omega.$$

Thus, in the following analysis we only consider $g = 0$. The weak formulation of the homogeneous Poisson problem consists of finding the weak solution $u \in H_0^1(\Omega)$ satisfying

$$(\nabla u, \nabla v) = (f, v), \tag{3}$$

for all test functions $v \in H_0^1(\Omega)$, where $(\cdot, \cdot)$ is the L^2 inner product in Ω.

Given the finite element space $V_h(\Omega) \subseteq H_0^1(\Omega)$, the approximate solution $u_h \in V_h(\Omega)$ of the discrete weak formulation satisfies

$$(\nabla u_h, \nabla v_h) = (f, v_h), \tag{4}$$

for all test functions $v_h \in V_h(\Omega)$.

Similarly, Neumann or mixed boundary conditions can also be specified on some part of the boundary. In this case, the spaces for the test functions and the solution in the weak formulation should be modified and the boundary terms are treated again as source terms, and details can be found in [26].

2.1. Domain discretization and notation

Let Ω be a closed and bounded subset of $\mathbb{R}^d$, for $d = 2, 3$. Let $J \geq 1$ and $k = 0, \dots, J$. For a given coarse quasi-uniform triangulation $\mathcal{T}_0$ in Ω, the mesh is uniformly refined once, to obtain the first level triangulation $\mathcal{T}_1$, and for $k \geq 2$ it is adaptively refined based on local error estimators. Then, for $k = 2 \dots, J$, nonuniform triangulations $\mathcal{T}_k$ cover the whole domain.

Let $\{\Omega^l\}_{l=1}^J$ be the collection of closed subdomains of Ω, which align with the edges of the elements in the nonuniform triangulation $\mathcal{T}_J$, such that their interior is covered only by a uniform triangulation $\mathcal{T}^l$ obtained after l refinements. Each Ω^l is such that $\text{int}(\Omega^i) \cap \text{int}(\Omega^j) = \emptyset$, if $i \neq j$, and $\Omega = \bigcup_{l=1}^J \Omega^l$. This means that $\{\Omega^l\}_{l=1}^J$ is a cover of Ω, and that the subdomains Ω^l either do not intersect or they do on a portion of their boundaries. Notice that each Ω^l can also be a union of disjoint subdomains. Also, notice that the coarse triangulation is uniformly refined, then $\Omega^0 = \emptyset$ and it has been excluded in the above. However, for generality we reintroduce it in the following.

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