ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A study of mimetic and finite difference methods for the static diffusion equation

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A R T I C L E I N F O

Article history:
Received 7 July 2017
Received in revised form 2 May 2018
Accepted 7 May 2018
Available online xxxx

Keywords:
Mimetic methods
Finite difference methods
Conservative methods
Steady-state diffusion equation
Boundary layer

A B S T R A C T

Two second-order finite difference methods in a staggered mesh to solve the static diffusion equation are proposed in this article. These methods were compared with a standard finite difference method and with two numerical schemes naturally established in staggered grids: mimetic method and conservative method. Also, mimetic discretization is presented in a formal manner. The methods were tested using different configurations, including boundary layers and heterogeneous media. The study shows that the two proposed finite difference methods produce numerical solutions that are comparable to those given by mimetic methods, in terms of rates of convergence and magnitude of the approximation error.

1. Introduction

The finite difference method for the discretization of boundary value problems has been one of the most used for decades because of its simplicity in the implementation, precision in the results and relative correspondence with the physics of the problem [1–3]. In order to preserve the second order approximations obtained at the internal nodes of the mesh used for the discretization of the physical medium, the standard finite difference method incorporates fictive nodes (*ghost points*) outside the physical domain. In addition, it discretizes the partial differential equation not only inside the medium but through the discrete domain, that is: inner nodes, nodes on the boundary and ghost points. In the latter, the discretization of the equation is not considered directly, but instead these ghost points appear in the discretization of both the boundary condition and the partial differential equation, although they do not appear explicitly in the final linear system of equations. To avoid the use of ghost points, one-side finite difference approximation has been proposed on the boundary but most of them produce numerical schemes with low order truncation error.

In the last two decades, a new group of numerical methods, initially based on finite difference, has been developed for the numerical resolution of boundary value problems. The support operator methods [4–6], later called mimetic methods [7–9], make up this group. The main feature of mimetic methods is that they discretize the basic operators (gradient, divergence, curl) and not the boundary value problem itself, preserving symmetry and satisfying a discrete version of Green–Stokes–Gauss theorem, which ensures compatibility between the partial differential equation and the boundary conditions. It has been reported that mimetic methods produce better results than standard finite difference [10–13], also, a quadratic convergence rate has been estimated in numerical tests and proved rigorously [10,11,14–16].

In this article, two methods in finite difference on the staggered mesh are proposed; both exhibit a second order in the discretization of the boundary value problem at all nodes of the mesh. For the nodes adjacent to the boundary where standard

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https://doi.org/10.1016/j.camwa.2018.05.004
0898-1221/

finite difference does not use a second order approximation, the first proposed method incorporates a set of nodes that allow to make a second order approximation. The second method uses ghost points to approximate the derivatives with second order finite difference at the nodes adjacent to the boundary.

This article is organized as follows. In Section 2 we describe the mathematical problem to be solved including the partial differential equation involved and the boundary conditions that are considered. In Section 3, the mimetic method is presented by showing the discretization of the operators gradient and divergence in the staggered uniform grid in 1D. We also show that the discretizations given by this method satisfy a semi-discrete version of Green–Stokes–Gauss theorem, and that the composition of the discrete operators gradient and divergence is consistent with Laplace equation inside the domain. Section 4 briefly describes the conservative method, which is a particular case of mimetic methods. Section 5 shows a standard finite difference method adapted to the staggered grid. We also propose two finite difference methods that use second order approximation to discretize the differential equation at all mesh points. In Section 6, some numerical results in homogeneous and heterogeneous media are shown. Finally, some conclusions are presented in Section 7.

2. Continuous model

The static diffusion equation is a second order elliptical partial differential equation which describes density fluctuations in a material undergoing diffusion [17]. In general, the static linear non-homogeneous diffusion equation is given by the following

$$-\nabla\cdot\Big(\mathbf{K}(\vec{x})\nabla u(\vec{x})\Big)=F(\vec{x}), \tag{1}$$

where $u(\vec{x})$ is the density of the diffusing material at a location $\vec{x}$ on the medium $[0,1]\times[0,1]$ considered in this study, $\mathbf{K}(\vec{x})$ is a diffusion tensor of the medium and $F(\vec{x})$ represents the source term. The symbols $\nabla\cdot$ and ∇ stand for the divergence and gradient operators, respectively. If $\mathbf{K}=\mathbf{I}$, the equation is called Poisson's equation. The negative sign in Eq. (1) is given by Fourier's law, which states that the flux of the diffusing material is proportional to the diffusion tensor and the negative gradient of the density [17].

In order to have a well posed boundary value problem, a mixed boundary condition will be imposed. In its general formulation, this condition may be written in the form

$$\alpha u(\vec{x})+\beta(\mathbf{K}(\vec{x})\nabla u(\vec{x})\cdot\vec{n})=g(\vec{x}), \tag{2}$$

where the coefficients α and β can be selected depending on the phenomenon to be modeled on the boundaries, and $\vec{n}$ represents the outward unit vector normal to the boundary. If $\alpha=0$, then a Neumann or flux boundary condition is prescribed on that boundary; if $\beta=0$, then a Dirichlet or imposed value on the boundary is prescribed; if both values are not null then a mixed (or Robin) boundary condition is considered.

Under certain assumptions, the boundary value problem with mixed boundary condition, has a unique solution $u(\vec{x})$ [17].

3. Mimetic method

The boundary value problem to be solved in this article was presented in Section 2. It consists of the partial differential equation (1) and the boundary conditions (2). In terms of the operators involved, the gradient and the divergence appear in (1), whereas in (2) only the gradient acts on the directional derivative.

The mimetic method used in this work is based on the second-order discretization proposed by Castillo and Yasuda [9,12] for the operators gradient and divergence. In contrast to standard finite difference methods, this numerical method discretizes the operators that compose the boundary value problem, instead of the partial differential equation itself. In correspondence with the physics of the problem, the mimetic method does not require neither to discretize the equation on the boundary to preserve the second order of approximation, nor to incorporate ghost points [11,12].

3.1. Staggered grid

3.1.1. One-dimensional case

Mimetic methods require a *non-uniform point distributed grid* [18] with special characteristics on the boundary of the domain. Let us assume that the domain is the unit interval $[0,1]$ and that we consider a uniform grid, so that we divide the interval in n blocks or segments of length $h=1/n$. The sides of the blocks of the mesh are given by the points $x_i=ih$, for $i=0,1,\ldots,n$, and the centers of the blocks are given by $x_{i+\frac{1}{2}}=(x_i+x_{i+1})/2$, for $i=0,1,\ldots,n-1$. Fig. 1 shows where the function $u(x)$ and the operators gradient (G) and divergence (D) will be evaluated.

The function $u(x)$ will be evaluated at the points denoted by black dots, "•", which correspond to the centers of the blocks and the endpoints of the domain. The gradient will be evaluated at the sides of the blocks denoted by x's, "×", and the divergence is evaluated at the circles, "◯".

Please cite this article in press as: G. Sosa Jones, et al., A study of mimetic and finite difference methods for the static diffusion equation, Computers and Mathematics with Applications (2018), https://doi.org/10.1016/j.camwa.2018.05.004.