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Analyzing the combined multi-waves polynomial solutions in a two-layer-liquid medium

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ABSTRACT

In this work, we investigate the behavior of mixed wave solutions in two-layer-liquid with dispersive waveguide. These combined multi-wave solutions were obtained in polynomial types for the (3 + 1)-dimensional Yu–Toda–Sasa–Fukuyama (YTSF) equation with variable coefficients. By virtue of the generalized unified method and symbolic computations, we formally derive polynomial solutions for this equation. The obtained solutions include multi-soliton solutions, multi-periodic solutions, and multi-elliptic solutions. Furthermore, the physical insight and the movement role of the waves in the two-layers are discussed and analyzed graphically for different selections of the arbitrary functions of the obtained solutions.

1. Introduction

Nonlinear evolution equations with variable coefficients (vcNLEEs) are very important to describe many phenomena in different branches of science such as fluid mechanics, biology, plasma physics, and nonlinear optics [1–5]. The most suitable strategy for understanding the dynamics of physical models related to vcNLEEs is to find their exact solutions. The explicit solutions of these equations, if available, simplify the verification of numerical algorithms their stability analysis.

Different approaches were proposed in the literature for calculating the exact solutions for vcNLEEs. These strategies include Adomian decomposition [6], the variational iteration [6,7], the reduced differential transform [8,9], the modified simple equation [10] and the extended simplest equation [11] methods.

The work in [12] discussed the behavior of mixed two-soliton rational solutions of the (3 + 1)-dimensional Yu–Toda–Sasa–Fukuyama equation with variable coefficients (vcYTSF) in two-layer liquid medium, or in an elastic medium, defined as

$$\left(u_t + \alpha(t)\, u_{xxz} + \beta(t)\, u\, u_z + \gamma(t)\, u_x\, \partial_x^{-1} u_z\right)_x + \delta(t)\, u_{yy} = 0, \tag{1}$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$, and $\delta(t)$ are arbitrary functions and the integral operator is defined as $\partial_x^{-1}(.) = \int(.)\, dx$.

The vcYTSF is widely used when $\alpha(t)$, $\beta(t)$, $\gamma(t)$, and $\delta(t)$ are constants and is denoted by YTSF. In this case, the YTSF equation is an extension of the Bogoyavlenskii–Schif equation [13,14]. The linear solitary wave solution of YTSF equation was firstly given using the strong symmetry [15]. The non-traveling wave solution was found using auto-Backlund transformation and the generalized projective Riccati equation method [15,16]. Moreover, some soliton-like solutions and periodic solutions for potential YTSF equation were obtained by Hirota's bilinear method, the tanh–coth method, exp-function method, homoclinic test approach and extended homoclinic test approach, respectively [17–20].

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The authors in [12] used the unified method (UM) [21–24] and the generalized unified method (GUM) [23,25–31] to find only single and double soliton rational solutions of Eq. (1), respectively. Furthermore, the propagation of semi-self similar waves solutions of the Eq. (1) that are mixed with many types of other wave solutions are shown in two-layers-liquid medium. Other useful works with significant results can be found in [32–37] and some of the references therein.

The motivation for this work is two-fold. We first aim to carry out a detailed investigation of the two-layer liquid medium governed by the (3+1)-dimensional YTSF equation with variable coefficients. Our second goal from this systematic study is to show that this two-layer model is rich of a variety of significant features which include multi-soliton solutions, multi-periodic solutions, and multi-elliptic solutions. We will achieve our goals by using the GUM. The GUM is powerful, reliable to handle models in different branches of science and it used a simple algorithms for finding multi- or combined wave solutions comparing with other methods. But it is applicable to vcNLEEs when these models are completely or partially integrable.

In this paper, we pay more attention to extend the work in [12] to find other types of solutions (combined multi-wave polynomial solutions) of Eq. (1) which they are impressive for physics and mathematics. We also concern ourselves with the physical behavior of these solutions. Moreover, we will discuss the obtained solutions graphically under the effect of the variable coefficients of the vcYTSF equation Eq. (1).

This paper is organized as follows: Section 2 is devoted to the application of GUM to the vcYTSF equation while conclusions are given in Section 3.

2. Combined multi-wave polynomial solutions of vcYTSF

In this section, we apply the GUM to find combined multi-wave polynomial solutions of vcYTSF when $N = 2$. Using the potential transformation $u(x, y, z, t) = v_x(x, y, z, t)$ in Eq. (1), and integrating both sides of the resulting equation with respect to x, we get

$$v_{xt} + \alpha(t)\, v_{xxxz} + \beta(t)\, v_x\, v_{xz} + \gamma(t)\, v_{xx}\, v_z + \delta(t)\, v_{yy} = 0, \tag{2}$$

where the integral constant is considered to be zero.

For double-wave solutions, we assume that

$$\begin{aligned} &v(x, y, z, t) = v_1(\xi_1, \xi_2) = p_0(t) + \sum_{m=1}^{n} \sum_{i_1+i_2=m} p_{i_1,i_2}(t)\, \phi_1^{i_1}(\xi_1)\, \phi_2^{i_2}(\xi_2), \\ &(\phi_1'(\xi_1))^p = \sum_{r=0}^{pk} b_r(t)\, \phi_1^r(\xi_1),\ (\phi_2'(\xi_2))^p = \sum_{r=0}^{pk} c_r(t)\, \phi_2^r(\xi_2),\ p = 1, 2, \end{aligned} \tag{3}$$

where $\xi_1 = \alpha_1 x + \alpha_2 y + \alpha_3 z + \int \alpha_4(t)\, dt$, $\xi_2 = \beta_1 x + \beta_2 y + \beta_3 z + \int \beta_4(t)\, dt$, α_l, β_l are arbitrary constants for $l = 1, 2, 3$ and $\alpha_4(t)$, $\beta_4(t)$, $p_0(t)$, $p_{i_1,i_2}(t)$, $b_r(t)$, and $c_r(t)$ are arbitrary functions.

2.1. Combined multi-wave polynomial solutions when $p = 1$

In this part we try to find multi-wave solution in the form of soliton wave solutions or periodic wave solutions.

When $p = 1$, the balance condition yields $n = k - 1$, $k > 1$ and the consistency condition gives rise to $k \le 3$ [24–29,38]. Thus, the solutions exist when $k = 2, 3$.

Case 1: When $k = 2$, $n = 1$.

In this case, we have

$$\begin{aligned} &v(x, y, z, t) = v_1(\xi_1, \xi_2) = p_0(t) + p_{1,0}(t)\, \phi_1(\xi_1) + p_{0,1}(t)\, \phi_2(\xi_2), \\ &\phi_1'(\xi_1) = b_0(t) + b_1(t)\, \phi_1(\xi_1) + b_2(t)\, \phi_1^2(\xi_1),\ \phi_2'(\xi_2) = c_0(t) + c_1(t)\, \phi_2(\xi_2) + c_2(t)\, \phi_2^2(\xi_2). \end{aligned} \tag{4}$$

By substituting from (4) into (2) and by equating the coefficients of ϕ_j, $j = 1, 2$ to be zero, we get a set of algebraic equations. By using any package of symbolic computations (such as the elimination method or other suitable solvable method with the aid of MATHEMATICA or MAPLE), we get

$$\begin{aligned} &p_{1,0}(t) = -\frac{6\,\alpha_1\, b_2(t)\,\alpha(t)}{\beta(t)},\quad p_{0,1}(t) = -\frac{6\,\beta_1\, c_2(t)\,\alpha(t)}{\beta(t)},\quad \gamma(t) = \beta(t), \\ &\alpha_4(t) = \frac{\alpha_1^3\,\beta_3\, R_1^2(t)\,\alpha(t)}{\beta_1} - \frac{\alpha_2^2\,\delta(t)}{\alpha_1},\quad \beta_4(t) = \frac{\beta_2^2\,\delta(t)}{\beta_1} - \beta_3\,\beta_1^2\, R_2^2(t)\,\alpha(t),\quad \alpha_3 = -\frac{\alpha_1\,\beta_3}{\beta_1}, \end{aligned} \tag{5}$$

where $R_1(t) = \sqrt{b_1^2(t) - 4\, b_0(t)\, b_2(t)}$ and $R_2(t) = \sqrt{c_1^2(t) - 4\, c_0(t)\, c_2(t)}$.

It remains to solve the auxiliary equations in (4). By a direct calculation, we get

$$\begin{aligned} &\phi_1(\xi_1) = -\frac{b_1(t) + R_1(t)\ \tanh(\frac{1}{2}\, R_1(t)\,\xi_1)}{2\, b_2(t)}, \\ &\phi_2(\xi_2) = -\frac{c_1(t) + R_2(t)\ \tanh(\frac{1}{2}\, R_2(t)\,\xi_2)}{2\, c_2(t)}, \end{aligned} \tag{6}$$