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A posteriori error analysis of the fully discretized time-dependent coupled Darcy and Stokes equations

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ABSTRACT

We present an a posteriori error analysis of the fully discretized time-dependent Darcy and Stokes equations, that models laminar fluid flow over a porous medium in two- or three-dimensional connected open domains which are coupled via appropriate matching conditions on the interface. The problem is discretized by the backward Euler scheme in time and finite elements in space.

1. Introduction

The coupled models find numerous practical applications, for example the flow of blood in arteries, diffusion and dispersion of pollutants in water, the flow with heat transfer in porous media, etc. These coupled problems involve several partial differential equations such as the Navier–Stokes equations, the Stokes equations, the Darcy's law in porous region and the free surface Navier–Stokes equations, [1–4]. The flow inside a deformable tube can be modeled using the linear elasticity equations coupled with the Navier–Stokes equations.

In this paper, we focus on a model for a liquid such as water flowing on a homogeneous porous ground, where the instationary Darcy and Stokes equations are coupled via appropriate matching conditions on the interface. This kind of problem is studied in [5] for the stationary case.

One important issue in the coupled Darcy–Stokes flow is the treatment of the interface condition, where the Stokes fluid meets the porous medium.

[3,6–14] considered a formulation based on the Beaver–Joseph–Saffman interface conditions, which was experimentally derived by Beavers and Joseph in [15]. For these Darcy–Stokes equations, the following interface conditions have been extensively studied and used in literature see [3,11] and [16, Sec. 4.5]

$$\boldsymbol{u}|_{\Omega_P} \cdot \boldsymbol{n} = \boldsymbol{u}|_{\Omega_F} \cdot \boldsymbol{n} \quad \text{and} \quad -p|_{\Omega_P}\boldsymbol{n} = \nu\partial_n\boldsymbol{u}|_{\Omega_F} - p|_{\Omega_F}\boldsymbol{n} \quad \text{on } \Gamma\times]0,T[, \tag{1}$$

the first and the second interface conditions ensure the mass conservation and the continuity of forces, respectively, across the interface Γ. However, in [8] the interface conditions refer to mass conservation, balance of normal forces in addition

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to the Beavers–Joseph–Saffman law, which yields the introduction of the trace of the porous media pressure as a suitable Lagrange multiplier.

Various numerical methods, such as finite element methods, mixed methods, discontinuous Galerkin methods and combinations of these methods, have been studied in the literature. For instance, finite element methods were studied in [3] and finite element methods coupled with mixed methods have been analyzed in [11]. Primal discontinuous Galerkin methods using broken Sobolev spaces are analyzed in [12], and they are coupled with mixed methods in [13].

The stability of the numerical methods for the evolutionary Stokes–Darcy problem was analyzed in different previous works see [17–19]. For example, [19] proposed an approach consisting of four methods that uncouple each time step into separate Stokes flow problem and Darcy flow problem, one is a parallel uncoupling method, while the three others uncouple sequentially. In [17] a second order and unconditionally stable method for the unsteady Stokes–Darcy problem is proposed. This method uncouples the surface from the groundwater flow by using the implicit–explicit combination of the Crank–Nicolson and Leapfrog methods for the discretization in time.

The basic discretization of this problem relies on the backward Euler scheme with respect to the time variable and on finite elements with respect to the space variables. The space discretization that we propose relies on the mortar element method, a domain decomposition technique introduced in [20] and [21]. We use a subdomain for the fluid and another one for the porous medium. On each subdomain, we consider a finite element discretization, relying on standard finite elements both for the Stokes problem and the Darcy equations. For the Stokes problem we use the Bernardi–Raugel finite element introduced in [22] and analyzed in [23] and for the Darcy problem we use the Raviart–Thomas finite element, see [24]. Besides, in [8] the same finite elements are employed for the Stokes and the Darcy domains, whereas the Lagrange multiplier on the interface is approximated by continuous piecewise linear elements.

Other choices of finite elements are possible. Indeed, the Raviart–Thomas element is the simplest div-conforming element and the Bernardi–Raugel element is the less expensive H^1-conforming finite element for the Stokes problem.

The a priori analysis of this problem has been recently published, see [25]. The aim of the present paper is to extend the investigation to the a posteriori analysis.

Several works have been done concerning the a posteriori analysis of parabolic type problems. Part of it (*cf*. [26–28]) deals only with the space discretization and provides appropriate error indicators for it. Another idea see [29–31], consists in establishing a full time and space variational formulation of the continuous problem and using a discontinuous Galerkin method for the discretization with respect to all variables. In this work, we follow a different approach that uncouples as much as possible the time and space errors, according to an idea presented in [32]. We introduce two different types of error indicators, one for the time discretization and other for the space discretization, and we prove upper and lower bounds for the error.

An outline of the paper is as follows:

- Section 2 is devoted to the description of the continuous, the time semi-discrete and the fully discrete problems. We recall their main properties and some standard a priori estimates.
- In Section 3, we perform the a posteriori analysis of the time discretization.
- In Section 4, three families of error indicators, related to the error on Ω_P, Ω_F and Γ, respectively, are proposed and the a posteriori analysis of the discrete problem is achieved.
- The adaptivity strategy is presented in Section 5.

2. The continuous, semi-discrete and discrete problems

The mortar element method has been used for handling curved boundaries. In order to avoid the techniques required for the treatment of the curved boundaries, we assume that the domain Ω is a polygonal in dimension ($d = 2$) or a polyhedral in dimension ($d = 3$) divided into two connected open sets Ω_P and Ω_F with Lipschitz-continuous boundaries, where the indices P and F stand for porous and fluid, respectively. Let $T > 0$ be a finite time. The fluid that we consider is viscous and incompressible and is governed by the Stokes equations:

$$\begin{cases} \partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \mathbf{grad} p = \boldsymbol{f} & \text{in} \quad \Omega_F \times]0,T[, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega_F \times]0,T[, \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}^0(\boldsymbol{x}) & \text{in} \quad \Omega_F \quad \text{at} \quad t = 0. \end{cases} \tag{2}$$

The porous medium is assumed to be rigid and saturated with the fluid, and governed by the Darcy equations:

$$\begin{cases} \partial_t \boldsymbol{u} + \alpha \boldsymbol{u} + \mathbf{grad} p = \boldsymbol{f} & \text{in} \quad \Omega_P \times]0,T[, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega_P \times]0,T[, \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}^0(\boldsymbol{x}) & \text{in} \quad \Omega_P \quad \text{at} \quad t = 0. \end{cases} \tag{3}$$

In problems (2) and (3) the unknowns are the velocity $\boldsymbol{u}$ and the pressure p of the fluid; the data are the distribution $\boldsymbol{f}$ which represents the external force and the initial velocity $\boldsymbol{u}^0$, while the parameters ν and α are positive constants, representing the viscosity of the fluid and the ratio of this viscosity to the permeability of the medium, respectively. We assume that α is a constant on Ω_P, which implies that the porous medium is homogeneous, see [33] and [34]. Concerning

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