Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

# Existence results and iterative method for solving a nonlinear biharmonic equation of Kirchhoff type

Quang A. Dang [a], Thanh Huong Nguyen [b,*]

[a] Centre for Informatics and Computing, VAST, 18 Hoang Quoc Viet, Cau Giay, Hanoi, Viet Nam
[b] College of Sciences, Thainguyen University, Thainguyen, Viet Nam

ARTICLE INFO

*Article history:*
Received 5 December 2017
Received in revised form 19 March 2018
Accepted 30 March 2018
Available online xxxx

*Keywords:*
Nonlinear biharmonic equation
Kirchhoff type equation
Existence and uniqueness of solution
Iterative method

ABSTRACT

In this paper, for investigating a boundary value problem for a nonlinear biharmonic equation of Kirchhoff type, we propose an efficient approach by the reduction of the problem to an operator equation for the nonlinear part of the equation. The result is that we have established the existence and uniqueness of a solution under some easily verified conditions. Also, we propose an iterative method for finding the solution. Some examples demonstrate the applicability of the theoretical results and the efficiency of the iterative method.



## 1. Introduction

In the paper, we consider the following nonlinear biharmonic boundary value problem (BVP):

$$\begin{aligned} &\Delta^2 u = M\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u + f(x,u), \quad x \in \Omega, \\ &\quad u = 0, \quad \Delta u = 0, \quad x \in \Gamma, \end{aligned} \tag{1.1}$$

where $\Omega$ is a connected bounded domain in $\mathbb{R}^K$ $(K \geq 2)$ with a smooth boundary $\Gamma$, $\Delta$ is the Laplace operator, $\Delta^2$ is the biharmonic operator, $\nabla u$ is the gradient of $u, f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $M : \mathbb{R}^+ \to \mathbb{R}$ are continuous functions. This problem describes the nonlinear static deflection of an elastic plate.

In one-dimensional case ($K = 1$), by using variational methods and some fixed point theorems in cones, the existence and multiplicity results for the problem for Eq. (1.1) with other boundary conditions are considered (see e.g. [1–4]). For a particular case, when $M(t) = t$ and $f = f(x)$ in [5] the authors studied the existence and uniqueness of a solution and proposed an iterative method for finding the solution.

In the multi-dimensional case $K > 1$, also by using the variational methods, many authors studied the existence and multiplicity of nontrivial solutions of the problem (1.1)(in the both cases when $M$ is a constant or not) (see e.g. [6–10]). It should be noticed that in the listed works, the function $f(x,u)$ is assumed with very strong assumptions, and *there are no examples of solutions to verify theoretical results of their existence*. Below we mention two typical works of them.

The first work is of Hu and Wang [7] concerned with the case when $M$ is a constant function, namely, the problem

$$\begin{aligned} &\Delta^2 u + c\Delta u = f(x,u), \quad x \in \Omega, \\ &u = 0, \quad \Delta u = 0, \quad x \in \Gamma, \end{aligned}$$

* Corresponding author.
*E-mail addresses:* dangquanga@cic.vast.vn (Q.A. Dang), nguyenthanhhuong2806@gmail.com (T.H. Nguyen).

https://doi.org/10.1016/j.camwa.2018.03.048
0898-1221/

where the space dimension $K > 4$, $c < \lambda_1$, $\lambda_1$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. By using a variant version of the mountain pass theorem they established the existence of nontrivial solutions for the problem under the assumption that the function $f(x,u)$ satisfies the following two kinds of hypotheses:

(A1) $f(x,t) \in C(\overline{\Omega} \times \mathbb{R})$; $f(x,t) \geq 0, \forall x \in \overline{\Omega}, t > 0$; $f(x,t) \equiv 0, \forall x \in \overline{\Omega}, t \leq 0$;
(A2) $\lim_{t\to 0^+} \frac{f(x,t)}{t} = p(x)$, $\lim_{t\to+\infty} \frac{f(x,t)}{t} = l$ uniformly a.e. $x \in \Omega$, where $0 \leq p(x) \in L^\infty(\Omega)$, $\|p(x)\|_\infty < \Lambda_1$ and $\Lambda_1 < l < +\infty$; $\Lambda_1$ is the first eigenvalue of $(\Delta^2 + c\Delta, H^2(\Omega) \cap H_0^1(\Omega))$

or

(B1) $f(x,t) \in C(\overline{\Omega} \times \mathbb{R})$; $f(x,0) = 0, \forall x \in \overline{\Omega}$; $f(x,t)t \geq 0, \forall (x,t) \in (\overline{\Omega} \times \mathbb{R})$;
(B2) $\lim_{t\to 0^+} \frac{f(x,t)}{t} = p(x)$, $\lim_{t\to+\infty} \frac{f(x,t)}{t} = l$ uniformly a.e. $x \in \Omega$, where $0 \leq p(x) \in L^\infty(\Omega)$, $\|p(x)\|_\infty < \Lambda_1$, $\Lambda_1 < l < +\infty$ and $l$ is not any of the eigenvalue of $(\Delta^2 + c\Delta, H^2(\Omega) \cap H_0^1(\Omega))$; $\Lambda_1$ is the first eigenvalue of $(\Delta^2 + c\Delta, H^2(\Omega) \cap H_0^1(\Omega))$.

In the case $M$ is not a constant function, in [10], the authors studied the existence of a positive solution for the problem (1.1). There it is assumed that

(H1) $M(t)$ is continuous and satisfies

$$M(t) > m_0, \quad \forall t > 0,$$

for some $m_0 > 0$. In addition, there exist $m' > m_0$ and $t_0 > 0$ such that

$$M(t) = m', \quad \forall t > t_0;$$

(H2) $f(x,t) \in C(\Omega \times \mathbb{R})$; $f(x,t) \equiv 0, \forall x \in \Omega, t \leq 0$; $f(x,t) \geq 0, \forall x \in \Omega, t > 0$;
(H3) $|f(x,t)| \leq a(x) + b|t|^p, \forall t \in \mathbb{R}$ and a.e. $x \in \Omega$, where $a(x) \in L^q(\Omega)$, $b \in \mathbb{R}$ and $1 < p < \frac{K+4}{K-4}$ if $K > 4$ and $1 < p < \infty$ if $K \leq 4$ and $\frac{1}{p} + \frac{1}{q} = 1$;
(H4) $f(x,t) = o(|t|)$ as $t \to 0$ uniformly for $x \in \Omega$;
(H5) There exists a constant $\Theta > 2$, $\overline{R} > 2$ such that

$$\Theta F(x,s) \leq sf(x,s), \quad \forall |s| \geq \overline{R},$$

where $F(x,s) = \int_0^s f(x,t)dt$.

It should be emphasized that the above mentioned works are of pure theoretical character. In these works the authors only proved the existence of solutions *without examples of existing solutions*.

In this paper, based on our idea in a previous paper [11] when studying the Neumann problem for a biharmonic type equation, we reduce the problem (1.1) to an operator equation for the right-hand side function. The idea of the reduction of boundary value problems for nonlinear fourth order differential equations were used by ourselves in the recent works [12–14]. Here, we prove that under some *assumptions, which are easily verified*, imposed on the functions $f$ and $M$, the operator is contractive. This ensures the existence and uniqueness of a solution for the original boundary value problem. Also, we establish the convergence of an iterative method for solving this problem. Some examples demonstrate the applicability of the obtained theoretical results and numerical experiments show the effectiveness of the iterative method.

## 2. Existence results

To investigate the problem (1.1) we shall associate it with a fixed point problem as follows.

For functions $\varphi(x) \in C(\overline{\Omega})$ consider the nonlinear operator defined by

$$(A\varphi)(x) = M\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u + f(x,u), \tag{2.1}$$

where $u(x)$ is a solution of the problem

$$\begin{aligned} &\Delta^2 u = \varphi(x), \ x \in \Omega, \\ &u = \Delta u = 0, \ x \in \Gamma. \end{aligned} \tag{2.2}$$

**Proposition 2.1.** *A function $\varphi(x)$ is a fixed point of the operator $A$, i.e., $\varphi(x)$ is a solution of the operator equation*

$$A\varphi = \varphi, \tag{2.3}$$

*where $A$ is the operator defined above, if and only if the function $u(x)$ determined from the boundary value problem (2.2) satisfies the problem (1.1)*