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# An integral equation method for the numerical solution of the Burgers equation

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ABSTRACT

We consider an initial–boundary value problem for the two-dimensional Burgers equation on the plane. This problem is reformulated by an equivalent integral equation on the Fourier transform space. For the solution of this integral equation, two numerical methods are proposed. One of these two methods is based on the properties of the Gaussian function, whereas the other one is based on the FFT algorithm. Finally, the Galerkin method with Gaussian basis functions is applied to the original initial–boundary value problem, in order to compare the performances of the proposed methods with a standard numerical procedure.

Some numerical examples are given to evaluate the efficiency of the proposed methods. The performances obtained from these numerical experiments promise that these methods can be applied effectively to more complex problems, such as the Navier–Stokes equation and the turbulence flows.



## 1. Introduction

Fluid dynamics problems arouse the interest of the scientific community for the great importance of their applications. They are difficult problems, whose analytical properties and approximation techniques are only partially known. The three-dimensional incompressible Navier–Stokes (NS) equation is one of the main open problems in Mathematics. Existence, smoothness and breakdown of the Navier–Stokes solutions on $\mathbb{R}^3$ and on $\mathbb{R}^3/\mathbb{Z}^3$ has been identified by the Clay Mathematics Institute [1] as one of the millennium problems.

The Burgers equation can be seen as a simplified fluid dynamics model and, from the scientific computation point of view, it can be used to setup computational tools in order to deal with more difficult problems as the Navier–Stokes equation. It was introduced by Bateman [2], who mentioned that this kind of equation was worthy of study and he gave its steady solutions. It was later studied by Burgers [3] as a mathematical model for turbulence, after whom such an equation is widely referred to as the Burgers equation. Nowadays, this model has been employed in a large variety of applications, such as dynamics of soil water, statistics of flow problems, mixing and turbulent diffusion, and so on, see [3,4] and [5] for details.

In this paper, we consider the following problem for $\boldsymbol{x} \in \mathbb{R}^2$ and $t > 0$:

$$\frac{\partial}{\partial t}\boldsymbol{u}(\boldsymbol{x},t) + \sum_{j=1,2} u_j(\boldsymbol{x},t)\frac{\partial}{\partial x_j}\boldsymbol{u}(\boldsymbol{x},t) = \nu \sum_{j=1,2} \frac{\partial^2}{\partial x_j^2}\boldsymbol{u}(\boldsymbol{x},t), \qquad (1)$$

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \qquad (2)$$

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$$\lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{0}, \tag{3}$$

where $\nu > 0$ is the viscosity, $\boldsymbol{u}_0 : \mathbb{R}^2 \to \mathbb{C}^2$ is the known initial solution, and $\boldsymbol{u} = (u_1, u_2) : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{C}^2$ is the unknown function. Note that (1) is the Burgers equation with viscosity $\nu$, (2) is the initial condition, and (3) is the boundary condition.

Several methods have been proposed for the numerical solution of the Burgers equation. For example: in [6] a meshless method for the computation of a numerical solution of unsteady coupled Burgers' equations is studied, in [7] a new numerical method based on the coupling between localized multiscale method and meshless method is developed for the 2D Burgers equation, in [8] the Tikhonov regularization is used to stabilize problems with a large Reynolds number; in [9] a stabilization technique for the 2D Burgers problem is presented, in [10], problem (1)–(3) is reduced to an equivalent problem on a bounded domain by introducing suitable absorbing conditions on the boundary of this domain, and the resulting problem is solved by a finite difference method; in [11] and in [12] the Burgers equation on a rectangular domain is discretized by an implicit finite-difference scheme; in [13] a finite element approach is proposed for the one-dimensional version of the Burgers equation on a bounded interval; in [14] a finite difference method is proposed for the non homogeneous Burgers equation on the line; in [15] the discrete Adomian decomposition method is used to solve the two-dimensional Burgers equation; in [16] an integral formulation of problem (1)–(3) is used to numerically prove the existence of exploding solutions at finite time; moreover, the same integral formulation has been studied in [17] to analyze its analytic properties, and in [18,19] to obtain an artificial boundary condition allowing the reduction of the problem (1)–(3) to an equivalent one on a bounded domain.

In this paper, we propose two methods for the numerical solution of problem (1)–(3) that are based on the integral formulation, that has been already used and studied in [16–18] and [19]. In particular, by using the Fourier transform, problem (1)–(3) can be rewritten as a nonlinear integral equation of convolution type. So, in one of the proposed methods, this integral equation is numerically solved by a standard FFT technique. In the other method, it is solved by taking advantage of the Gaussian function properties.

The results obtained by these two methods are compared with the results obtained from a direct solution of problem (1)–(3) by using the Galerkin method [20] with Gaussian basis functions. This comparison is based on numerical experiments.

In Section 2, we describe the integral formulation of problem (1)–(3). In Section 3, we propose two methods for the solution of the integral formulation of problem (1)–(3). In Section 4, we describe the Galerkin method with Gaussian basis functions. In Section 5, we show numerical experiments, where the results obtained by the three methods described in the previous sections are compared. Finally, in Section 6 we give some conclusions and future developments of the present study.

## 2. Integral equation

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^+$ be the set of positive real numbers, $\mathbb{C}$ be the set of complex numbers, $\iota$ be the imaginary unit. Let $k, d \in \mathbb{N}$, we denote with $\mathbb{R}^d$, $\mathbb{C}^k$ the $d$-dimensional real Euclidean space and the $k$-dimensional complex Euclidean space, respectively. Let $\boldsymbol{x}, \ \boldsymbol{y} \in \mathbb{R}^d$, we denote with $\boldsymbol{x}^T$ the transpose of $\boldsymbol{x}$, with $\boldsymbol{x}^T\boldsymbol{y}$ the scalar product of $\boldsymbol{x}$ and $\boldsymbol{y}$, and with $|\boldsymbol{x}|$ the Euclidean norm of $\boldsymbol{x}$. Let $\mathcal{L}^2$ be the space of functions that are square integrable and let $\mathcal{L}^1$ be the space of functions whose absolute values are integrable.

Let $\boldsymbol{f} : \mathbb{R}^d \to \mathbb{C}^k, \boldsymbol{f} \in \mathcal{L}^1(\mathbb{R}^d, \mathbb{C}^k) \cap \mathcal{L}^2(\mathbb{R}^d, \mathbb{C}^k)$, the Fourier transform of $\boldsymbol{f}$, that will be denoted with $\widehat{\boldsymbol{f}}$, is defined as

$$\widehat{\boldsymbol{f}}(\boldsymbol{\xi}) = \left(\frac{1}{2\pi}\right)^{d/2} \int_{\mathbb{R}^d} e^{-\iota\boldsymbol{\xi}^T\boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}) d\boldsymbol{x}, \qquad \boldsymbol{\xi} \in \mathbb{R}^d, \tag{4}$$

and the inverse Fourier transform of $\boldsymbol{f}$, that will be denoted with $\check{\boldsymbol{f}}$, is

$$\check{\boldsymbol{f}}(\boldsymbol{x}) = \left(\frac{1}{2\pi}\right)^{d/2} \int_{\mathbb{R}^d} e^{\iota\boldsymbol{\xi}^T\boldsymbol{x}} f(\boldsymbol{\xi}) d\boldsymbol{\xi}, \qquad \boldsymbol{x} \in \mathbb{R}^d. \tag{5}$$

We consider problem (1)–(3). We suppose that, for each $t > 0$, the following conditions hold:

$$\boldsymbol{u}(\cdot, t),\ \frac{\partial \boldsymbol{u}}{\partial t}(\cdot, t),\ \frac{\partial \boldsymbol{u}}{\partial x_j}(\cdot, t),\ u_j\frac{\partial \boldsymbol{u}}{\partial x_j}(\cdot, t) \in \mathcal{L}^1(\mathbb{R}^2, \mathbb{C}^2) \cap \mathcal{L}^2(\mathbb{R}^2, \mathbb{C}^2),\ j = 1, 2, \tag{6}$$

$$\lim_{|\boldsymbol{x}|\to\infty} \frac{\partial \boldsymbol{u}}{\partial x_j}(\boldsymbol{x}, t) = \boldsymbol{0}, \qquad j = 1, 2. \tag{7}$$

By denoting with $\widehat{\boldsymbol{g}}(\boldsymbol{\xi}, t)$ the Fourier transform of a function $\boldsymbol{g}(\boldsymbol{x}, t)$ with respect to variable $\boldsymbol{x} \in \mathbb{R}^2$ and by using standard arguments about the Fourier transform theory, we have that for $j = 1, 2,\ \boldsymbol{\xi} \in \mathbb{R}^2$ and $t > 0$ a generic fixed time

$$\begin{aligned}
\widehat{\frac{\partial \boldsymbol{u}}{\partial t}}(\boldsymbol{\xi}, t) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-\iota\boldsymbol{\xi}^T\boldsymbol{x}} \frac{\partial}{\partial t} \boldsymbol{u}(\boldsymbol{x}, t) d\boldsymbol{x} = \frac{\partial}{\partial t} \widehat{\boldsymbol{u}}(\boldsymbol{\xi}, t), \\
\widehat{\frac{\partial \boldsymbol{u}}{\partial x_j}}(\boldsymbol{\xi}, t) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-\iota\boldsymbol{\xi}^T\boldsymbol{x}} \frac{\partial}{\partial x_j} \boldsymbol{u}(\boldsymbol{x}, t) d\boldsymbol{x} = \iota\xi_j \widehat{\boldsymbol{u}}(\boldsymbol{\xi}, t), \\
\widehat{\frac{\partial^2 \boldsymbol{u}}{\partial x_j^2}}(\boldsymbol{\xi}, t) &= \iota\xi_j \widehat{\frac{\partial \boldsymbol{u}}{\partial x_j}}(\boldsymbol{\xi}, t) = -\xi_j^2 \widehat{\boldsymbol{u}}(\boldsymbol{\xi}, t),
\end{aligned}$$