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Two-grid finite element method and its a posteriori error estimates for a nonmonotone quasilinear elliptic problem under minimal regularity of data

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ABSTRACT

In this paper, we propose and analyze a two-grid finite element method for a class of quasilinear elliptic problems under minimal regularity of data in a bounded convex polygonal $\Omega \subset \mathbb{R}^2$, which can be thought of as a type of linearization of the nonlinear system using a solution from a coarse finite element space. With this technique, solving a quasilinear elliptic problem on the fine finite element space is reduced into solving a linear problem on the fine finite element space and solving the quasilinear elliptic problem on a coarse space. Convergence estimates in the H^1-norm are derived to justify the efficiency of the proposed two-grid algorithm. Moreover, we propose a natural and computationally efficient residual-based a posteriori error estimator of the two-grid finite element method for this nonmonotone quasilinear elliptic problem and derive the global upper and lower bounds on the error in the H^1-norm. Numerical experiments are provided to confirm our theoretical findings.

1. Introduction

The purpose of this paper is to study the convergence and the a posteriori error estimates of the two-grid finite element method for the following nonmonotone quasilinear Dirichlet problem

$$\begin{cases} -\nabla\cdot(a(x,u)\nabla u)+f(x,u)=0, & x\in\Omega,\\ u=0, & x\in\Gamma, \end{cases} \tag{1.1}$$

where Ω is an open bounded convex polygonal in $\mathbb{R}^2$ and Γ is its boundary. The smoothness requirements on a and f will be given in detail in Section 2.

There are some important numerical results available for (1.1) with $f(x,u)=f(x)$. The uniqueness of the classical and weak solutions of (1.1) were established in [1] and [2], respectively. Douglas and Dupont [3] proved an optimal rate of convergence of the finite element approximation for solving (1.1). The result in [3] was generalized in [4] to any smooth uniformly positive definite matrix $a(x,u)=(a_{ij}(x,u))_{i,j=1}^2$. Milner [5] considered mixed finite element method for (1.1) and showed an optimal rate of convergence in the L^p-norm. Similar results were obtained in [6]. Gudi and Pani [7], Bi and Ginting [8,9] studied the a priori error estimates, two-grid algorithm and the a posteriori error estimates of discontinuous

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Galerkin method for (1.1), respectively. Chatzipantelidis, Ginting and Lazarov [10], Bergam, Mghazli and Verfürth [11], Bi and Ginting [12] studied the a priori error estimates and a posteriori error estimates of the finite volume element method of (1.1), respectively.

Note that in these papers cited above, in order to develop the existence, uniqueness of approximation solution, the a priori and a posteriori error estimates, the function a and the exact solution of (1.1) are assumed to have higher smoothness. For example, it was assumed that a belongs to C^2 and $u \in W^{2,2+\epsilon}(\Omega)$, $\epsilon > 0$, or $u \in H^2(\Omega)$, see [3–5,7–9,13] for details. Recently, Casas and Dhamo [14] have proved that C^2 regularity of a is not necessary to establish the existence and a priori error estimates of finite element approximation of (1.1) in the convex and non-convex planar polygonal or polyhedral domains and only requires a local Lipschitz property of a. Moreover, Casas and Dhamo [14] proved that the exact solution of (1.1) with minimal regularity of data is only in $H^{3/2}(\Omega)$, see Section 2 for details.

Two-grid finite element methods based on two finite element spaces on one coarse and one fine grid were first introduced by Xu [13]-[15] for the nonsymmetric linear and nonlinear elliptic problems and these articles derived the convergence estimates to justify the efficiency of these algorithms. Later on, the two-grid methods were investigated for solving many other problems, for instance, Xu and Zhou [16] for eigenvalue problems, Axelsson and Layton [17] for nonlinear elliptic problems, Dawson, Wheeler and Woodward [18] for finite difference method for nonlinear parabolic equations, Utnes [19] for Navier–Stokes equations, Marion and Xu [20] for evolution equations, Wu and Allen [21], Chen and Chen [22] for mixed finite element method to solve reaction–diffusion equations, Bi and Ginting [23] for the finite volume element method for the nonlinear elliptic problems and Bi and Ginting [8] for the discontinuous Galerkin finite element method for (1.1). Congreve and Houston [24] for the discontinuous Galerkin finite element method for quasi-Newtonian fluid flow problem, Chen and Liu [25] for the finite volume element method for the nonlinear parabolic problems. These past works have given an indication on the viability of the two-grid methods as an efficient technique for solving nonlinear problems of various type.

Note that in the papers mentioned above on the two-grid algorithm for various nonlinear problems, the coefficient function and the exact solutions of the nonlinear problems are assumed to have higher smoothness. For example, it was assumed that the exact solution $u \in W^{2,2+\epsilon}(\Omega)$, $\epsilon > 0$, [13,23]; $u \in W^{2,\infty}(\Omega)$ [18] and $u \in H^2(\Omega)$ [20], see those paper for details. The relaxation from higher regularity to lower regularity on the coefficient function and the exact solutions of the nonlinear problems is a technical point of interest since the later is general and the former holds only on very favorable cases.

In this paper, we propose and analyze the two-grid technique to solve the nonmonotone quasilinear elliptic boundary value problem (1.1) under minimal regularity of data and $u \in H^{3/2}(\Omega)$. The discretization is based on one coarse and one fine conforming linear finite element spaces, S_H and S_h, respectively, where H and h are the grid sizes for the coarse grid and fine grid, respectively. With this technique, solving a quasilinear elliptic problem on the fine space S_h is reduced into solving a linear problem on the fine space S_h and solving the quasilinear elliptic problem on a coarse space S_H. This means that solving a quasilinear elliptic problem is not much more difficult than solving one linear problem, since $\dim S_H \ll \dim S_h$ and the work for solving the quasilinear problem on the coarse grid is relatively small. Under the assumption that the mesh parameter is sufficiently small, we show the convergence rate of the proposed two-grid algorithm in the H^1-norm for $u \in H^{3/2}(\Omega)$. This assumption that the mesh parameter is sufficiently small is reasonable, which guarantees the existence and uniqueness of the finite element approximation of (1.1), see [14] for details.

A posteriori error estimates of the finite element method have been studied extensively in the past several decades and some important results have been achieved. We refer the reader to monographs [26–29] and [30] and references therein for an extensive survey of the vast amount of research in this field, many of which were concentrated on linear problems.

Many authors studied the a posteriori error estimates of numerical solution of (1.1) under the higher regularity assumptions on a, u, and $f(x, u) = f(x)$. Bergam, Mghazli and Verfürth [11], Bi and Ginting [12] studied the a posteriori error estimates of finite volume element method of (1.1). Bi and Ginting [9] developed the a posteriori error estimates of discontinuous Galerkin finite element method for (1.1) under the assumption that $a \in C_b^2(\Omega \times \mathbb{R})$, where $C_b^2(\Omega \times \mathbb{R})$ is the class of twice continuously differentiable functions on $\Omega \times \mathbb{R}$ such that all derivatives of a_{ij} up to and including second order are bounded in $\Omega \times \mathbb{R}$. Under the assumption $u \in C^{1,\nu}(\overline{\Omega})$ for some $0 < \nu \le 1$ or $u \in W^{2,\infty}(\Omega)$, Demlow [31] provided two types of computationally efficient residual-based pointwise a posteriori error estimators for gradients of piecewise linear finite element approximations of (1.1). Liu et al. [32] derived the global postprocessing-based a posteriori error estimators in the H^1- and L^2-norms of the rectangular finite element approximations of (1.1) in the case that $a(x, u) = (a_{ij}(x, u))_{i,j=1}^2$ is a bounded uniformly positive definite matrix and $a_{ij} \in C_b^2(\Omega \times \mathbb{R})$. Verfürth [33] presented a general framework to establish a posteriori error estimates for the finite element solution of nonlinear elliptic problems $F(u) = 0$ with somewhat restrictive regularity imposed on F, i.e., $F \in C^1(X, Y^*)$ or $F \in C(X, Y^*)$ and F is a monotone operator. Here X and Y are two Banach spaces, and $*$ denotes the dual of a Banach space. Application of this framework to (1.1) in section 6 in [33] requires $a \in C^1(\Omega \times \mathbb{R})$, and yields upper and lower bounds on the error only in the $W^{1,p}$-norm with $2 < p < 4$, which does not cover the important a posteriori error estimates in the H^1-norm.

In [34], we presented an analysis of a posteriori error estimates of two-grid finite element method for the general second-order nonlinear elliptic problems under the regularity assumption that $u \in H_0^1(\Omega) \cap W^{2,2+\epsilon}(\Omega)$ for some $\epsilon > 0$. In this paper, we further to propose a natural and computationally easy residual-based a posteriori error estimator of the two-grid finite element method for (1.1) under minimal regularity of data and $u \in H^{3/2}(\Omega)$. Compared with the assumptions on the data in [14], in order to make sense of the residual-based a posteriori error estimator on finite elements, we only require