ARTICLE IN PRESS
Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

# Convergence of the finite difference scheme for a general class of the spatial segregation of reaction–diffusion systems

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ARTICLE INFO

*Article history:*
Received 22 April 2017
Received in revised form 20 February 2018
Accepted 10 March 2018
Available online xxxx

*Keywords:*
Free boundary
Obstacle-like problems
Reaction–diffusion systems
Finite difference method

ABSTRACT

In this work we prove convergence of the finite difference scheme for equations of stationary states of a general class of the spatial segregation of reaction–diffusion systems with $m \geq 2$ components. More precisely, we show that the numerical solution $u_h^l$, given by the difference scheme, converges to the $l^{th}$ component $u_l$, when the mesh size $h$ tends to zero, provided $u_l \in C^2(\Omega)$, for every $l = 1, 2, \ldots, m$. In particular, our proof provides convergence of a difference scheme for the multi-phase obstacle problem.


## 1. Introduction

### 1.1. The setting of the problem

In recent years there have been intense studies of spatial segregation for reaction–diffusion systems. The existence of spatially inhomogeneous solutions for competition models of Lotka–Volterra type in the case of two and more competing densities has been considered in [1–7]. The aforementioned segregation problems led to an interesting class of multi-phase obstacle-like free boundary problems. These problems have growing interest due to their important applications in the different branches of applied mathematics. To see the diversity of applications we refer [8–10] and the references therein.

Nowadays, the theory of the one- and two-phase obstacle-like problems (elliptic and parabolic versions) is well-established and for a reference we address to the books [11,12] and references therein. For two-phase problems the interested reader is also referred to the recent works [13,14].

There is a vast literature devoted to the numerical analysis of one-phase obstacle-like problems, and we refer some of well-known papers [15–18]. For the numerical treatment of the two-phase problems we refer to the works [19–24].

The present work concerns to prove the convergence of the difference scheme for a certain class of the spatial segregation of reaction–diffusion system with $m$ components.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a connected and bounded domain with smooth boundary and $m$ be a fixed integer. We consider the steady-states of $m$ competing species coexisting in the same area $\Omega$. Let $u_i(x)$ denote the population density of the $i^{\text{th}}$ component with the internal dynamic prescribed by $F_i(x, u_i)$.

We call the $m$-tuple $U = (u_1, \ldots, u_m) \in (W^{1,2}(\Omega))^m$, *a segregated state* if

$$u_i(x) \cdot u_j(x) = 0, \text{ a.e. for } \quad i \neq j, \ x \in \Omega.$$

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https://doi.org/10.1016/j.camwa.2018.03.025
0898-1221/

The problem amounts to

$$\text{Minimize } E(u_1,\dots,u_m)=\int_\Omega \sum_{i=1}^m\left(\frac{1}{2}|\nabla u_i|^2+F_i(x,u_i)\right)dx, \tag{1}$$

over the set

$$S=\{(u_1,\dots,u_m)\in (W^{1,2}(\Omega))^m : u_i\geq 0, u_i\cdot u_j=0, u_i=\phi_i \quad \text{on} \quad \partial\Omega\},$$

where $\phi_i\in H^{\frac{1}{2}}(\partial\Omega)$, $\phi_i\cdot\phi_j=0$, for $i\neq j$ and $\phi_i\geq 0$ on the boundary $\partial\Omega$.

We assume that

$$F_i(x,s)=\int_0^s f_i(x,v)dv,$$

where $f_i(x,s):\Omega\times\mathbb{R}^+\to\mathbb{R}$ is Lipschitz continuous in $s$, uniformly continuous in $x$ and $f_i(x,0)\equiv 0$.

**Remark 1.** Functions $f_i(x,s)$'s are defined only for non negative values of s (recall that our densities $u_i$'s are assumed non negative); thus we can arbitrarily define such functions on the negative semiaxis. For the sake of convenience, when $s\leq 0$, we will let $f_i(x,s)=-f_i(x,-s)$. This extension preserves the continuity due to the conditions on $f_i$ defined above. In the same way, each $F_i$ is extended as an even function.

**Remark 2.** We emphasize that for the case $f_i(x,s)=f_i(x)$, the assumption is that for all $i$ the functions $f_i(x,s)$ are nonnegative and uniformly continuous in $x$. Also for simplicity, throughout the paper we shall call both $F_i(x,u_i)$ and $f_i(x,u_i)$ internal dynamics.

We would like to point out that the only difference between our minimization problem (1) and the problem discussed in [2], is the sign in front of the internal dynamics $F_i$. In our case, the plus sign of $F_i$ allows to get rid of some additional conditions, which are imposed in [2, Section 2]. Those conditions are important to provide coercivity of a minimizing functional in [2]. But in our case the above given conditions together with convexity assumption on $F_i(x,s)$, with respect to the variable $s$ are enough to conclude $F_i(x,u_i(x))\geq 0$, which in turn implies coercivity of a functional (1).

In order to speak on the local properties of the population densities, let us introduce the notion of multiplicity of a point in $\Omega$.

**Definition 1.** The multiplicity of the point $x\in\overline{\Omega}$ is defined by:

$$m(x)=\text{card}\,\{i: measure(\Omega_i\cap B(x,r))>0, \forall r>0\},$$

where $\Omega_i=\{u_i>0\}$.

For the local properties of $u_i$ the same results as in [2] with the opposite sign in front of the internal dynamics $f_i$ hold. Below, for the sake of clarity, we write down these results from [2] with appropriate changes.

**Lemma 1** (*Proposition 6.3 in [2]*). *Assume that $x_0\in\Omega$, then the following holds:*

(1) *If $m(x_0)=0$, then there exists $r>0$ such that for every $i=1,\dots,m$;*

$$u_i\equiv 0 \quad on \quad B(x_0,r).$$

(2) *If $m(x_0)=1$, then there are $i$ and $r>0$ such that in $B(x_0,r)$*

$$\Delta u_i=f_i(x,u_i), \qquad u_j\equiv 0 \quad for\ j\neq i.$$

(3) *If $m(x_0)=2$, then there are $i,j$ and $r>0$ such that for every $k$ and $k\neq i,j$, we have $u_k\equiv 0$ and*

$$\Delta(u_i-u_j)=f_i(x,(u_i-u_j))\chi_{\{u_i>u_j\}}-f_j(x,-(u_i-u_j))\chi_{\{u_i<u_j\}} \text{ in } B(x_0,r).$$

**Lemma 2** (*Theorem 5.1 in [2]*). *For every minimizer $(u_1,\dots,u_m)\in S$ to the functional (1), the following inequality holds*

$$\Delta\left(u_l(x)-\sum_{p\neq l}u_p(x)\right)\leq f_l(x,u_l),$$

*for all $l=1,2,\dots,m$.*

Next, we state the following uniqueness theorem due to Conti, Terracini and Verzini, by observing that in our case the plus sign in front of $F_i$ requires convexity condition on $F_i(x,s)$ rather than concavity condition given in [2].

**Theorem 1** (*Theorem 4.2 in [2]*). *Let the functional in minimization problem (1) be coercive and moreover each $F_i(x,s)$ is convex in the variable $s$, for all $x\in\Omega$. Then, the problem (1) has a unique minimizer.*