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# A note on the stability parameter in Nitsche's method for unfitted boundary value problems

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## HIGHLIGHTS

- A theoretical error analysis of Nitsche's method on unfitted grids is presented.
- It is shown that error bounds deteriorate for specific cut configurations.
- Examples with large and diverging discretization errors are demonstrated.
- The effect on gradients in small cut elements is investigated.
- An overview of resolutions is presented.

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## ABSTRACT

Nitsche's method is a popular approach to implement Dirichlet-type boundary conditions in situations where a strong imposition is either inconvenient or simply not feasible. The method is widely applied in the context of unfitted finite element methods. Of the classical (symmetric) Nitsche's method it is well-known that the stabilization parameter in the method has to be chosen sufficiently large to obtain unique solvability of discrete systems. In this short note we discuss an often used strategy to set the stabilization parameter and describe a possible problem that can arise from this. We show that in specific situations error bounds can deteriorate and give examples of computations where Nitsche's method yields large and even diverging discretization errors.



## 1. Introduction

We consider the discretization of an unfitted Poisson problem. The problem domain is described separately from the encapsulating mesh, on which a finite element basis is defined. We consider the restriction of this finite element space with respect to the problem domain. Such an approach is used in many methods which are similar in virtue, *e.g.*, the fictitious domain method [1], the cut finite element method (CutFEM) [2–5], the finite cell method (FCM) [6–12], immersogeometric analysis [13–15], the unfitted discontinuous Galerkin method (UDG) [16,17], the extended finite element method (XFEM) [18–23], and several others. To impose essential boundary conditions, many of these methods apply some version of the classical Nitsche's method [24].

This method requires stabilization to preserve the coercivity of the bilinear operator. Without additional stabilization of cut elements with *e.g.*, ghost penalty terms [2,25] –which is customary for methods referred to as CutFEM –the corresponding

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stabilization parameter depends on the shape and size of the cut elements, *i.e.*, it depends on the position of the geometry relative to the computational mesh. A typical choice which is sufficient to provide unique solvability of discrete problems, is to choose the stabilization parameter as an elementwise constant and proportional to the ratio between the surface measure of the intersection between an element and the boundary, and the volume measure of the intersection of the same element and the domain. As this ratio can become arbitrarily large, the stabilization parameter is not generally bounded.

In the mathematical literature, *e.g.*, [2,3], unfitted Nitsche formulations are generally supplemented by the aforementioned additional stabilization to bound the stabilization parameter in order to prove properties of the method. However, it is our opinion that the importance of this has not been sufficiently addressed in the literature. Moreover, the method has effectively (and successfully) been applied without additional stabilization in the engineering literature, see *e.g.*, [10–12]. With this note, we detailedly treat the possible problem that can occur when no additional measures are taken to bound the stabilization parameter and aim to provide a disclaimer for directly applying the classical form of Nitsche's method to immersed problems. To this end, we discuss and analyze the method and demonstrate that it can lead to poor results in the discretization error when the geometry intersects the computational domain such that large values of the stabilization parameter are required. We also investigate different situations where degenerated geometry configurations yield satisfactory and unsatisfactory results and provide a possible interpretation. Furthermore, we review alternative formulations and possible modifications and stabilizations of the method.

Section 2 of this note introduces a model problem, followed by the Nitsche's method under consideration in Section 3. In Section 4 we carry out a simple error analysis of the method in a norm which is natural to the formulation. This analysis reveals the possible large discretization errors for unfortunate cut configurations. In Section 5 we give examples of bad cut configurations and show how these can lead to large and even degenerating discretization errors. Finally, we list alternative approaches to impose boundary conditions and variants of the classical Nitsche's method to circumvent this possible problem in Section 6.

## 2. The model problem

As a model for more general elliptic boundary value problems, we consider the Poisson problem with inhomogeneous Dirichlet boundary data posed on an open and bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$:

$$-\Delta u = f \quad \text{in } \Omega, \tag{1a}$$
$$u = g \quad \text{on } \partial\Omega. \tag{1b}$$

We assume that the properties of the domain $\Omega$ imply that a shift theorem of the form:

$$\|u\|_{H^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}, \tag{2}$$

holds whenever $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$, *e.g.*, when $\partial\Omega$ is smooth or $\Omega \subset \mathbb{R}^2$ is a convex domain with piecewise $C^2$ boundary [26]. In this note $\lesssim$ is used to denote an inequality with a constant that is independent of the mesh. This problem has the standard well-posed weak formulation: Find $u \in H^1_g(\Omega)$, such that:

$$\int_\Omega \nabla u \nabla v \, dx = \int_\Omega f v \, dx \quad \text{for all } v \in H^1_0(\Omega). \tag{3}$$

In this formulation $H^1_0(\Omega)$ and $H^1_g(\Omega)$ are the standard Sobolev spaces with corresponding homogeneous and inhomogeneous boundary data. For simplicity, we assume that Dirichlet boundary conditions are posed on the whole boundary of $\Omega$. However, the results presented in this work extend to the case where Dirichlet boundary data are only prescribed on a part of the boundary as demonstrated in Section 5.

## 3. A Nitsche-based unfitted finite element method

We consider a (triangular or rectilinear) shape regular *background* discretization $\widetilde{\mathcal{T}}$ which encapsulates the domain $\Omega$. To each element $T \in \widetilde{\mathcal{T}}$ we associate the local mesh size $h_T = \text{diam}(T)$ and define the global mesh size by $h = \max_{T\in\widetilde{\mathcal{T}}} h_T$. We define the *active* mesh $\mathcal{T}$ by:

$$\mathcal{T} = \{T \in \widetilde{\mathcal{T}} : T \cap \Omega \neq \emptyset\}, \tag{4}$$

and set $\widetilde{\Omega} = \bigcup_{T\in\mathcal{T}} T$, denoting the union of all elements from the active mesh. The set of geometric entities is illustrated in Fig. 1. On the active mesh $\mathcal{T}$ we define the finite dimensional function space:

$$V_h = \{v \in C^0(\widetilde{\Omega}) : v|_T \in P_k(T), T \in \mathcal{T}\}, \tag{5}$$

consisting of continuous piecewise polynomials of order $k$. A simple prototype formulation for a Nitsche-based unfitted finite element method is to find $u_h \in V_h$, such that:

$$a_h(u_h, v_h) = l_h(v_h) \quad \forall\, v_h \in V_h, \tag{6}$$