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A source identification problem in a time-fractional wave equation with a dynamical boundary condition

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ABSTRACT

We focus on a inverse source problem in a partial differential equation containing a fractional derivative in time of the order $1 < \beta < 2$. The equation is accompanied with a non-standard boundary condition which consists of the classic flux term and the dynamical time-fractional derivative term on the one part of the boundary. To determine both the solution of the equation and the source term, a measurement in a form of a integral over space domain is considered. Using a time-discretization and Rothe's method, we prove an existence of the strong solution in the suitable functional spaces, and the error estimate is established; moreover, the uniqueness is addressed. The results are supported by some numerical experiments.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be bounded with the Lipschitz boundary Γ and $T > 0$, we deal with the equation

$$\left(g_{2-\beta} * \partial_{tt} u(x)\right)(t) - \Delta u(x,t) = h(t) f(x), \qquad x \in \Omega,\ t \in (0,T), \tag{1.1}$$

where the kernel $g_{2-\beta}$ is defined as

$$g_{2-\beta}(t) = \frac{t^{1-\beta}}{\Gamma(2-\beta)}, \qquad t > 0,\ 1 < \beta < 2,$$

and $*$ is a convolution operator on the positive half-line

$$(k * v)(t) = \int_0^t k(t-s) v(s)\, ds.$$

Eq. (1.1) is often called the fractional wave equation, since it contains the Caputo fractional derivative of order $\beta \in (1,2)$ defined as

$$\partial_t^\beta u(x,t) = \left(g_{2-\beta} * \partial_{tt} u(x)\right)(t),$$

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Notice that for $\beta = 1$ or $\beta = 2$ the equation becomes the classical diffusion or wave equation, respectively. Eq. (1.1) is accompanied with the following initial and boundary conditions

$$\begin{aligned} u(x,0) &= u_0(x), \quad x \in \Omega, \\ \partial_t u(x,0) &= v_0(x), \quad x \in \Omega, \\ u(x,t) &= 0, \qquad (x,t) \in \Gamma_D \times (0,T), \\ -\left(g_{2-\beta} * \partial_{tt} u(x)\right)(t) - \nabla u(x,t) \cdot \boldsymbol{\nu} &= \sigma(x,t), \; (x,t) \in \Gamma_N \times (0,T), \end{aligned} \tag{1.2}$$

where we assume $\Gamma_D \cap \Gamma_N = \varnothing$, $\overline{\Gamma}_D \cup \overline{\Gamma}_N = \Gamma$, $|\Gamma_D| > 0$, and $\boldsymbol{\nu}$ is a outer normal vector on Γ. The boundary condition we consider on the part of the boundary Γ_N is often called the dynamical boundary condition as it contains the time derivative of the solution.

The inverse source problem (ISP) we are interested in here consists of finding the couple (u, h). To accomplish this it is necessary to possess an additional measurement. This is assumed to have the following form

$$\int_\Omega u(x,t)\omega(x)\,\mathrm{d}x = m(t), \qquad t \in [0,T], \tag{1.3}$$

where the function ω is solely space dependent. Frequently, it is assumed, ω is with compact support in Ω, then the measurement (1.3) can be interpreted as the weighted average over the sub-domain of Ω, see [1].

The fractional wave equation can be found in models of diffusive waves propagation in viscoelastic materials, see [2,3]. The existence and uniqueness of the direct Cauchy problem is addressed in [4]. Obtaining the fundamental solution by the means of Laplace transform is studied in [5]. Eq. (1.1) without the source term is studied in [6], where authors derived explicit expression for solution through the corresponding Green's functions in terms of Fox functions and provided probabilistic interpretation of the equation in one-dimensional case. The equation without the source term is also studied in [7,8]. Other interesting studies concerning the direct problem for fractional wave equation are [9–11]. The identification of the source term is a well discussed problem in inverse problems (IPs), both in parabolic and hyperbolic settings. Recognition of the solely space-dependent part of source is studied in [1,12–18]. The problems concerning the identification of the time-dependent source cf. [1,19–21]. The inverse source problems for equations containing the fractional derivative became popular in recent years. Many articles deal with the fractional diffusion equation (cf. [22–25,4,26–28]). In [29,30] a reconstruction of time-dependent source part in a fractional integrodifferential wave equation has been studied. Both article results are based on the use of the Banach fixed point theorem. The measurement in [29] is in the form of time trace at the point inside the domain. In [30] the source term and the convolution kernel were identified from two measurements in the form of integral over the subdomain. Both articles consider the zero Dirichlet boundary condition. The hyperbolic equation accompanied with the dynamical boundary condition for 1D space can model a viscoelastic rod with a mass attached to its free tip, see [31]. According to [32] such a boundary condition can also occur in modeling a flexible membrane with boundary affected by vibration only in a region. In [33] the dynamical boundary condition is derived including the influence of the heavy frame in the modeling of small vertical oscillation of flexible membrane. The direct problem for the fractional diffusion equation with the dynamical boundary condition was studied in [34]. The dynamical boundary condition in (1.2) containing the fractional derivative is a generalization of the dynamical boundary condition containing the classical derivative as in [35]. The boundary condition with a convolution term containing the solution can be found in [36].

The paper is organized as follows. In the second section we introduce some notation used in the article and state the variational formulation of our problem. We reformulate our problem into the direct one by applying the measurement on Eq. (1.1) and gaining the second equation for the couple (u, h). In the third section the uniqueness of the inverse problem is addressed in the appropriate spaces. In the fourth section the time discretization is introduced, the existence of the solutions along each of the slices is shown, and the a priori estimates are proven. We then define the Rothe functions and state the existence theorem in which we prove the convergence of those functions to the solution of our problem. The error estimate is presented in the fifth section. In the last part we present a couple of numerical experiments. The solution is calculated for various values of time step and different measurement functions. We also present calculation with a possible treatment of the noisy data.

2. Reformulation of problem

In this short section we state notations used in the article and we introduce the variational formulation our problem.

For the standard inner product on $L^2(\Omega)$ and its induced norm we use the notation $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. In case of boundary space $L^2(\Gamma)$ we use the similar notation, specifically $(\cdot,\cdot)_\Gamma$ and $\|\cdot\|_\Gamma$, the analogue is used for boundary space on Γ_N. For the Banach space X with the norm $\|\cdot\|_X$ and the abstract function $w : [0,T] \to X$ we define the space $C([0,T], X)$ as the set of continuous functions w endowed with the norm $\max_{t\in[0,T]} \|\cdot\|_X$. Moreover, the set of p-integrable abstract functions w for $p > 1$ furnished with the norm $\left(\int_0^T \|\cdot\|_X^p \,\mathrm{d}t\right)^{\frac{1}{p}}$ is denoted as $L^p((0,T), X)$, cf. [37]. The dual space of X is denoted by X^*. In the whole text the generic positive constants depending on the data will be notated as C, ε and C_ε, where ε is considered to be a small one and $C_\varepsilon = C\left(\frac{1}{\varepsilon}\right)$ a large one. The same notation can occur for the different values of those constants in the same discussion.

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