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Dynamics of a diffusive vaccination model with nonlinear incidence ☆

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ABSTRACT

A recent paper Xu et al. (2018) studied the global stability and traveling wave solution of a vaccination model with nonlinear incidence. Furthermore, in this paper, we first study the local stability of this model, and then establish the uniform persistence result. Our results are a supplement to above paper.

1. Introduction

Recently, Xu et al. [1] considered the following diffusive vaccination model with nonlinear incidence:

$$\begin{cases} S_t(x,t) = d_1\triangle S(x,t) + \Lambda - \sigma\beta_1 f(I(x,t))S(x,t) - (\mu+\gamma)S(x,t), \\ V_t(x,t) = d_2\triangle V(x,t) + \gamma S(x,t) - \sigma\beta_2 f(I(x,t))V(x,t) - \mu V(x,t), \\ I_t(x,t) = d_3\triangle I(x,t) + \sigma\beta_1 f(I(x,t))S(x,t) + \sigma\beta_2 f(I(x,t))V(x,t) - \mu I(x,t), \end{cases} \tag{1.1}$$

for $x \in \Omega, t > 0$. Here $S(x,t)$, $V(x,t)$ and $I(x,t)$ represent the number of compartments of susceptible, vaccinated and infectious individuals at position x and time t, respectively. Ω is a connected, bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. $\triangle$ is the Laplacian operator, and $d_i > 0$ $(i = 1, 2, 3)$ are the diffusion rates. The parameters Λ, σ, β_1, μ, γ and β_2 are positive constants. The function $f(I) = \frac{I}{1+I}$.

The initial conditions of system (1.1) are

$$S(x,0) = S_0(x) \geq 0, \quad V(x,0) = V_0(x) \geq 0, \quad I(x,0) = I_0(x) \geq 0, \quad \forall x \in \overline{\Omega}, \tag{1.2}$$

and the Neumann boundary conditions are

$$\frac{\partial S}{\partial \nu}(x,t) = \frac{\partial V}{\partial \nu}(x,t) = \frac{\partial I}{\partial \nu}(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.3}$$

where $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative on $\partial\Omega$.

According to the threshold value $\mathcal{R}_0$, Xu et al. [1] established the results of the global stability of all equilibria for system (1.1). Furthermore, they studied the existence and nonexistence of traveling wave solutions of system (1.1). However, McCluskey and Yang [2] pointed out that the Lyapunov functional for the interior equilibrium may not be defined when

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the state variables are zero or close to zero. Thus, it is necessary to discuss the uniform persistence to ensure the Lyapunov functionals are well-defined. Notice that Xu et al. [1] did not discuss the uniform persistence of system (1.1)–(1.3). Hence, in this paper, we further investigate the uniform persistence and local stability of equilibria of system (1.1)–(1.3) as a supplement of Xu et al. [1].

This paper is organized as follows. In Section 2, we give some previous results from Xu et al. [1]. In Section 3, we establish the local stability of all equilibria. In Section 4, we explore the uniform persistence. Conclusions can be found in Section 5.

2. Preliminary results

In this section, we give some results from Xu et al. [1]. System (1.1) admits a disease-free equilibrium $E_0(S^0, V^0, 0)$, where

$$S^0 = \frac{\Lambda}{\mu+\gamma}, \quad V^0 = \frac{\Lambda\gamma}{\mu(\mu+\gamma)}.$$

The basic reproduction number of system (1.1) is

$$\mathcal{R}_0 = \frac{\beta_1\sigma\Lambda}{\mu(\mu+\gamma)} + \frac{\beta_2\sigma\Lambda\gamma}{\mu^2(\mu+\gamma)}.$$

When $\mathcal{R}_0 > 1$, system (1.1) has a unique disease equilibrium $E^*(S^*, V^*, I^*)$, where

$$S^* = \frac{\Lambda(1+I^*)}{\sigma\beta_1 I^* + (\mu+\gamma)(1+I^*)}, \quad V^* = \frac{\Lambda\gamma(1+I^*)^2}{(\sigma\beta_1 I^* + (\mu+\gamma)(1+I^*))(\sigma\beta_2 I^* + \mu(1+I^*))},$$

and I^* is a real positive root of the following equation:

$$a_2(I^*)^2 + a_1 I^* + a_0 = 0,$$

where

$$\begin{aligned}
a_2 &= -\mu(\mu^2 + \mu\gamma + \sigma^2\beta_1\beta_2 + \mu\sigma\beta_1 + \gamma\sigma\beta_2 + \mu\sigma\beta_2),\\
a_1 &= (\sigma^2\beta_1\beta_2 + \gamma\sigma\beta_2 + \mu\sigma\beta_1)\Lambda - \mu(2\mu\gamma + \sigma\beta_1\mu + 2\mu^2 + \mu\sigma\beta_2 + \gamma\sigma\beta_2),\\
a_0 &= (\beta_1\mu + \beta_2\gamma)\sigma\Lambda - \mu^2(\mu+\gamma).
\end{aligned}$$

Let $X := C(\overline{\Omega}, \mathbb{R}^3)$ be the Banach space with the supremum norm $\|\cdot\|_X$. Define $X^+ := C(\overline{\Omega}, \mathbb{R}^3_+)$, then (X, X^+) is a strongly ordered space.

Theorem 2.1. *For any given initial data $\varphi \in X^+$, system (1.1)–(1.3) has a unique solution $\mathbf{u}(\cdot, t, \varphi)$ on $[0, +\infty)$, and further the solution semiflow $\Phi(t) := \mathbf{u}(\cdot, t) : X^+ \to X^+$, $t \geq 0$, has a global compact attractor in X^+.*

Theorem 2.2.

(i) *If $\mathcal{R}_0 \leq 1$, then the disease-free equilbrium E_0 of system (1.1) is globally asymptotically stable;*
(ii) *If $\mathcal{R}_0 > 1$, then the disease equilibrium E^* of system (1.1) is globally asymptotically stable.*

3. Local stability

In this section, we discuss the local stability of equilibria for system (1.1). We have the following results.

Theorem 3.1.

(i) *When $\mathcal{R}_0 < 1$, the disease-free equilibrium E_0 of system (1.1) is locally asymptotically stable;*
(ii) *When $\mathcal{R}_0 > 1$, the disease equilibrium E^* of system (1.1) is locally asymptotically stable.*

Proof. Linearizing system (1.1) at E_0, we get

$$\frac{\partial Z(x,t)}{\partial t} = D\triangle Z(x,t) + \mathcal{B}_1 Z(x,t),$$

where $D = \mathrm{diag}(d_1, d_2, d_3)$ and

$$\mathcal{B}_1 = \begin{pmatrix} -(\mu+\gamma) & 0 & -\sigma\beta_1 S^0 \\ \gamma & -\mu & -\sigma\beta_2 V^0 \\ 0 & 0 & \sigma\beta_1 S^0 + \sigma\beta_2 V^0 - \mu \end{pmatrix}.$$

Thus, we can obtain the following characteristic polynomial

$$|\lambda I + Dl^2 - \mathcal{B}_1| = 0,$$