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An energy method of fundamental solutions for solving the inverse Cauchy problems of the Laplace equation

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A B S T R A C T

The accuracy of the method of fundamental solutions (MFS) is heavily influenced by the distribution of source points. One often locates the source points along an offset to the problem boundary or a circle with a fixed radius. In this paper we propose an energy regularization technique to choose the source points and weighting factors in the numerical solution of the inverse Cauchy problems for the Laplace equation in arbitrary domain. An inequality is derived, which is a criterion to pick up the source points and weighting factors. This new technique can improve the accuracy of the numerical solution than the MFS with the distribution of source points using a fixed offset. Some numerical tests confirm that the energy MFS (EMFS) has a good stability and accuracy, and the computational cost is cheap.

1. Introduction

The method of fundamental solutions (MFS) is one of the meshless numerical methods that is popularly used in the engineering computations. In the MFS the trial solution is a linear combination of fundamental solutions, which automatically satisfy the governing equation, and the coefficients in the linear combination are determined from the boundary conditions. The MFS is easy for numerical implementation and can avoid the integrations on the boundary. The MFS has been used to solve the Laplace equation [1–3]. Karageorghis et al. [4] have given a comprehensive review of the applications of the MFS to the inverse problems.

However, the MFS has a serious drawback that the resulting linear equations system may become highly ill-conditioned when the number of source points is increased or when the distances of source points are increased [5,6], which needs to reduce the condition number by searching a suitable distribution of source points. Usually, before the use of the MFS a suitable regularization is required [7,8].

Tsai et al. [9] have proposed a numerical procedure to locate the source points of the MFS, wherein the higher condition numbers and smaller errors are observed when the source points are located farther in a proper way. Young et al. [10] have proposed a modified method of fundamental solutions for solving the Laplace equation, which implements the singular fundamental solutions to evaluate the solutions, and it can locate the source points on the physical boundary.

The research on the distribution of source points is an important issue in the MFS [11–14]. Alves [15] has proposed the source points along the discrete normal direction with a possible local criterion to define the distance to the boundary. In

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the sense of the equilibrated matrix method, Liu [16] has chosen the source points solved from a set of uncoupled nonlinear algebraic equations.

The inverse Cauchy problems of the Laplace equation have a broad applications. In a practical solution of the inverse Cauchy problem it is highly ill-posed; hence, the numerical solution does not depend continuously on the Cauchy data, and a small error in the given data may lead to an incorrect solution. Owing to its extremal ill-posedness in nature, we must tackle the inverse Cauchy problems with stable numerical algorithms, which have been carried out by using different numerical methods, to name a few, the boundary element method [17–19], the modified collocation Trefftz method [20,21], the method of fundamental solutions (MFS) [7], and the Fourier regularization method [20,22].

This paper first proposes an energy method of fundamental solutions (EMFS) for the numerical solution of the inverse Cauchy problems for the Laplace equation in arbitrary domain. In the EMFS, an energy regularization technique is employed to choose the source points in the MFS and to determine the weighting factors in the new energy bases. As a criterion, an inequality is derived to pick up the source points and weighting factors. The proposed scheme is mathematically simple, numerically stable and costless, and can effectively improve the accuracy of the numerical solution of the traditional MFS.

The remaining portions of the present paper are arranged as follows. In Section 2 we describe the inverse Cauchy problem of the Laplace equation, and the fundamental solutions are given. The main results are shown in Section 3, where we derive an energy equation and multiply the fundamental solutions by weighting factors to satisfy the energy equation. An inequality is derived, such that we can develop two algorithms in Section 4 to choose the source points in the EMFS. The numerical tests are given in Section 5. Finally, the conclusions are drawn in Section 6.

2. The inverse Cauchy problem

The purpose of this paper is to develop a novel yet simple method to choose the source points of the MFS, according to the concept of energy which is generated from the governing equation and the boundary conditions, to reduce the ill-conditioning behavior and the sensitivity to the source points in the MFS. Consider the inverse Cauchy problem of the Laplace equation:

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \tag{1}$$

$$u(\rho,\theta) = h(\theta), \ \ 0 \le \theta \le \beta\pi, \tag{2}$$

$$u_n(\rho,\theta) = g(\theta), \ \ 0 \le \theta \le \beta\pi, \tag{3}$$

where $h(\theta)$ and $g(\theta)$ are known functions and $\beta \le 1$. The inverse Cauchy problem is to seek an unknown boundary function $f(\theta)$ on $\Gamma_2 := \{r = \rho(\theta), \beta\pi < \theta < 2\pi\}$ under the over-specified data on $\Gamma_1 := \{r = \rho(\theta), 0 \le \theta \le \beta\pi\}$, where $\rho(\theta)$ denotes the boundary shape of the problem domain $\Omega := \{(r,\theta) \mid r < \rho(\theta),\ 0 \le \theta \le 2\pi\}$. In Eq. (3), $u_n(\rho,\theta)$ can be computed by [21]

$$u_n(\rho,\theta) = \eta(\theta)\left[\frac{\partial u(\rho,\theta)}{\partial \rho} - \frac{\rho'}{\rho^2}\frac{\partial u(\rho,\theta)}{\partial \theta}\right], \tag{4}$$

where

$$\eta(\theta) = \frac{\rho(\theta)}{\sqrt{\rho^2(\theta) + \rho'(\theta)^2}}. \tag{5}$$

In the MFS the trial solution at the field point $\mathbf{z} = (r\cos\theta, r\sin\theta)^{\mathrm{T}}$ is expressed as a linear combination of the fundamental solutions $U(\mathbf{z}, \mathbf{s}_j)$:

$$u(\mathbf{z}) = \sum_{j=1}^{n} c_j U(\mathbf{z}, \mathbf{s}_j), \ \ \mathbf{s}_j \in \Omega^c, \tag{6}$$

where n is the number of source points, c_j are unknown coefficients to be determined, $\mathbf{s}_j$ are the source points, and Ω^c is the complementary set of Ω. For the Laplace equation (1) we have the fundamental solutions:

$$U(\mathbf{z}, \mathbf{s}_j) = \ln r_j, \ \ r_j = \|\mathbf{z} - \mathbf{s}_j\|, \tag{7}$$

which is singular when the source point $\mathbf{s}_j$ approaches to the field point $\mathbf{z}$.

3. Energy equation and new bases

Before embarking the analysis of the energy method of fundamental solutions (EMFS) in a linear space, we require the following result.

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