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Numerical contour integral methods for unsteady Stokes equations☆

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ABSTRACT

The unsteady Stokes equations are semi-discretized in space to obtain a system of linear time-invariant differential–algebraic equations (DAEs), i.e., the unsteady discrete Stokes equations. The solution to unsteady discrete Stokes equations is represented as an integral along a smooth curve Γ in the complex plane with singularities of the integrand located on the left of and not too close to the curve Γ. Truncated quadrature rules based on the sinc function are then employed to evaluate the solution. This results in a number of complex linear systems to solve, leading to major expense in practical implementation. Constraint preconditioners are proposed to work with the Krylov subspace methods for solving those complex linear systems. Numerical examples illustrate that the numerical contour integral methods are more effective than the time-stepping methods. In addition, the constraint preconditioners significantly improve the behavior of Krylov subspace methods for solving the involved complex linear systems.

1. Introduction

Consider the numerical solution of unsteady Stokes equations modeling "low-speed" incompressible viscous flow as follows:

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} - \nu \nabla^2 \vec{u} + \nabla p = \vec{f} & \text{in } \Omega \times [t_0, t_1], \\ \nabla \cdot \vec{u} = 0 & \text{in } \Omega \times [t_0, t_1], \end{cases} \tag{1.1}$$

where $\vec{u}$ is the velocity of fluid. p is the pressure of fluid. $\vec{f}$ is a given external force. $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is an open bounded domain. $\nu > 0$ is the kinematic viscosity. A boundary value problem is adding conditions to the system (1.1) on boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ as

$$\vec{u} = \vec{v} \text{ on } \partial\Omega_D, \quad \nu \frac{\partial \vec{u}}{\partial \vec{n}} - \vec{n}p = 0 \text{ on } \partial\Omega_N,$$

where $\vec{n}$ is the outward-pointing normal to the boundary. Semi-discretization in space of system (1.1) leads to a system of differential–algebraic equations (DAEs), i.e., the unsteady discrete Stokes equations:

$$\left(\mathcal{B}\frac{\mathrm{d}}{\mathrm{d}t} + \mathcal{A}\right)\mathbf{x} := \left\{\begin{pmatrix} H & 0 \\ 0 & 0 \end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}t} + \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}\right\}\begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} := \mathbf{b}, \ \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1.2}$$

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where H and $A \in \mathbb{R}^{n\times n}$ are Hermitian positive definite representing velocity mass matrix and discrete diffusion, respectively. $B^T \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times n}$ are full column rank and full row rank representing discrete gradient and negative discrete divergence, respectively. $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{p} \in \mathbb{R}^m$ are the discrete velocity and pressure. $\mathbf{f} \in \mathbb{R}^n$ and $\mathbf{g} \in \mathbb{R}^m$ are forcing and boundary terms. $\mathbf{x}_0 \in \mathbb{R}^{n+m}$ is the initial data.

The most frequently used methods for unsteady discrete Stokes equations (1.2) are time-stepping methods including Runge–Kutta methods [1] and linear multi-step methods [1]. The basic idea of time-stepping methods is to adopt temporal discretization to (1.2) on a prescribed time-level-sequence, then the evaluation of solution $\mathbf{x}$ is needed on each time level. Therefore, a small time-step-size always leads to a large number of evaluations of solution $\mathbf{x}$, thus making the workload of time-stepping methods increasing intensively. In order to overcome the disadvantages of time-stepping methods with small time-step-size, we employ the numerical contour integral methods instead based on efficient numerical Laplace inversion [2,3].

The paper is organized as follows. In Section 2, we state and analyze the numerical contour integral methods for unsteady discrete Stokes equations. In Section 3, we propose the constraint preconditioner for complex linear systems involved in the numerical contour integral methods. In Section 4, we list the numerical results to show the effectiveness of numerical contour integral methods and the proposed constraint preconditioner. In Section 5, we give some concluding remarks.

Notation: Denote $\imath = \sqrt{-1}$ as the imaginary unit. Denote $\mathbb{R}_-$ as the negative real axis. Denote $\Re(\cdot)$, $\Im(\cdot)$ as the real part and imaginary part of a number or a matrix.

2. Numerical contour integral method

In this section, we focus on numerical methods. Hence, we assume the existence of Laplace transform, Laplace inversion and other integrals without addressing the convergence conditions in the sequel. The Laplace transform of a function $\mathbf{v}$ is defined as

$$\widehat{\mathbf{v}} \equiv \widehat{\mathbf{v}}(s) = \int_0^\infty \mathbf{v}(t) e^{-st}\, \mathrm{d}t.$$

By taking the Laplace transform on both sides of the unsteady discrete Stokes equations (1.2), we have

$$\mathcal{B}(s\widehat{\mathbf{x}} - \mathbf{x}_0) + \mathcal{A}\widehat{\mathbf{x}} = \widehat{\mathbf{b}},$$

where $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{b}}$ are the Laplace transforms of $\mathbf{x}$ and $\mathbf{b}$, respectively. If the complex number s is not a generalized eigenvalue of the matrix pencil $(\mathcal{A}, -\mathcal{B})$, we obtain

$$\widehat{\mathbf{x}} = \widehat{\mathcal{A}_C}^{-1}\widetilde{\mathbf{b}} = (s\mathcal{B} + \mathcal{A})^{-1}\widetilde{\mathbf{b}} = \begin{pmatrix} sH + A & B^T \\ B & 0 \end{pmatrix}^{-1} \widetilde{\mathbf{b}} \quad \text{with} \quad \widetilde{\mathbf{b}} = \widehat{\mathbf{b}} + \mathcal{B}\mathbf{x}_0.$$

The Laplace inversion of $\widehat{\mathbf{x}}$ leads to the expression of $\mathbf{x}$, i.e.,

$$\mathbf{x} = \frac{1}{2\pi\imath}\int_{\sigma-\imath\infty}^{\sigma+\imath\infty} e^{st}\widehat{\mathbf{x}}\,\mathrm{d}s \quad \text{with} \quad \sigma > \sigma_0,$$

where σ_0 is the convergence abscissa. The above formula is known as the Bromwich integral. In order to evaluate $\mathbf{x}$ numerically, two properties should be guaranteed: first, the factor $\widehat{\mathbf{x}}$ is analytic in much of the complex plane besides the half-plane $\mathrm{Re}(s) > \sigma_0$; second, the factor e^{st} is analytic and decays rapidly as $\mathrm{Re}(s) \to -\infty$ along the integral curve. An idea to obtain these properties is to deform the Bromwich line $\mathrm{Re}(s) = \sigma$ into a contour Γ that is better suited for numerical computation, i.e.,

$$\mathbf{x} = \frac{1}{2\pi\imath}\int_\Gamma e^{st}\widehat{\mathbf{x}}\,\mathrm{d}s.$$

The contour Γ should be a positively oriented curve that encloses both the spectrum of matrix pencil $(\mathcal{A}, -\mathcal{B})$ and the singularities of $\widetilde{\mathbf{b}}$. This deforming idea is originated in [4,5] during the mid-1950s. In order to determine the curve Γ properly, we need to locate the spectrum of matrix pencil $(\mathcal{A}, -\mathcal{B})$ as in the following theorem.

Theorem 2.1. *The spectrum of matrix pencil $(\mathcal{A}, -\mathcal{B})$ is a subset of the strictly negative real axis, i.e.,*

$$\mathrm{sp}(\mathcal{A}, -\mathcal{B}) \subset \mathbb{R}_- \setminus 0.$$

Proof. First, we consider a special case, i.e., $H = I$. For a complex number $s \in \mathbb{C}$, if the shifted matrix $sI + A$ is invertible, the determinant of generalized characteristic matrix $s\mathcal{B} + \mathcal{A}$ of the matrix pencil $(\mathcal{A}, -\mathcal{B})$ is given by

$$\det(s\mathcal{B} + \mathcal{A}) = \det\begin{pmatrix} sI + A & B^T \\ B & 0 \end{pmatrix} = \det(sI + A)\det(-B(sI + A)^{-1}B^T).$$

If the complex number satisfies $\Re(s) \geq 0$, since A is Hermitian positive definite and B^T is full column rank, we have $\det(s\mathcal{B} + \mathcal{A}) \neq 0$. Hence, the spectrum of matrix pencil $(\mathcal{A}, -\mathcal{B})$ belongs to the left-half complex plane. Furthermore, we

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