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On an equivalent representation of the Green's function for the Helmholtz problem in a non-absorbing impedance half-plane

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ABSTRACT

The Green's function associated with the Helmholtz problem in a non-absorbing impedance half-plane can be expressed as an integral form. In the non-absorbing case, the presence of surface waves presents a challenge in order to obtain accurate approximations. In this work, we present an equivalent representation for this Green's function, expressed as a sum of analytical terms and bounded integrals. The resulting representation is numerically stable and it can be estimated by any well known robust integration rule for bounded intervals. We provide a detailed description of the equivalent representation and we validate it with numerical experimentation.

1. Introduction

Several problems in engineering can be modeled as the scattering of an incident acoustic field due to a local perturbation of a half-plane. One such problem is, for instance, the determination of electromagnetic modes of two-dimensional photonic crystals, where the photonic crystal is assumed to be covered by a substrate which is modeled as a half-plane [1,2]. Another situation where this problem appears is in the computation of harbor resonances in marine hydraulics, where the sea is assumed to fill a locally perturbed half-plane and the perturbation is given by the harbor geometry [3–5]. Another example is the propagation of elastic wave modes above the ground due to blasting and drilling operations in a mine. Here, the mine can be considered to be the local perturbation of a half-plane [6].

When considering a time-harmonic decomposition for acoustic waves, i.e., an explicit time dependency of the form $e^{-i\omega t}$, with ω denoting the angular frequency, the problem of acoustic sound propagation above ground can be reduced to the estimation of a spatial total field $u_T(\mathbf{x})$ (decomposed into a known incident field plus an unknown scattered field), satisfying the following impedance (or Robin) boundary condition (over the boundary of the locally perturbed half-plane):

$$\partial_n u_T(\mathbf{x}) = i\beta\kappa\, u_T(\mathbf{x}), \tag{1}$$

where ∂_n denotes the partial derivative in the normal direction (that is assumed to be outwardly directed), $\beta \in \mathbb{C}$ denotes the acoustic admittance of the boundary and $\kappa = \omega/c > 0$ is the wave number (with c being the sound speed). When the

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real part of the admittance constant is positive, the boundary has an absorbing-energy property, implying that the scattered field has the standard Sommerfeld acoustic behavior as $|\mathbf{x}| \to +\infty$ (cf. [7]). Existence and uniqueness results of this problem can be found in [8] and integral representations in [9]. When β is purely imaginary, and such that $\mathrm{Im}(\beta) < 0$, the boundary is said to have a non-absorbing property since it allows the propagation of surface waves. In such case, the scattered field has a different behavior near the surface. In [10], an equivalent Sommerfeld-type radiation condition is given explicitly for the scattered field and also existence and uniqueness results are proven for the solutions. In [11], an integral representation is presented and a boundary element formulation is defined over the local perturbation. A more detailed description can be found likewise in [12].

The main problem when dealing with such an integral representation is that the Green's function, which is given in terms of an integral form, needs to be approximated accurately. When considering the non-absorbing case, certain values of the involved parameters make the numerical approximation of the associated Green's function unstable, generating several difficulties in the implementation of the integral representation (cf. [12]). In [11] an Inverse Fast Fourier Transform (IFFT) numerical approximation for the non-absorbing case is proposed. The method is numerically stable and the authors validate it by considering several numerical experiments. The principal disadvantage of the IFFT method is the large number of samples required to obtain an accurate approximation for the Fourier transform, making it an inefficient approximation when a large number of Green's function evaluations is required, for instance, when considering its integral representation or when approximating it by the Boundary Element Method (BEM). In [12], an alternative approximation is proposed, which considers different approaches depending on the far and near field behaviors of the Green's function, and utilizes integral identities to treat singularities and numerical stability problems. Faster approximations than the IFFT approach are possible to obtain with the given representations, however at a trade-off in accuracy. Also, the treatment of several expressions is required for its implementation, making it difficult in practice. More recently, in [13] a hybrid method between the Sommerfeld integral and the method of images is proposed, which is claimed to be efficient numerically, however, the non-absorbing case is not treated therein.

The main contribution of this work is the development of a novel and equivalent representation for the Green's function related to the Helmholtz impedance problem in a half-plane, allowing the propagation of surface waves into infinity. This novel representation is numerically stable and has several advantages. On one hand, the involved integral terms are defined over bounded intervals. Such domains depend only on distances in terms of the source and observation points, which has a considerable advantage in the BEM implementation. On the other hand, the bounded nature of said integral terms makes it possible to estimate them by using any well known quadrature rule or numerical method for integrals defined over bounded domains. The equivalent representation can be easily extended to other impedance cases, for instance, to absorbing boundaries.

The remainder of this paper is organized as follows. First we present the direct scattering problem due to a local perturbation of the impedance half-plane. We provide also the integral representation (in terms of the Green's function) and the weak boundary formulation that is satisfied by the outgoing scattered field. Second, we describe the Green's function associated with the impedance problem. Third, we introduce the equivalent representation for the non-absorbing case, giving an extended and detailed explanation of its deduction. We also provide explicit formulas for the computations of the partial derivatives and we discuss the extension to general impedance-type problems. Fourth, we perform numerical experiments validating the accuracy of the proposed approximation. Finally, we conclude and discuss potential future lines of work.

2. The locally perturbed half-plane problem

Let $\mathbb{R}^2_+ := \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 > 0\}$ be the upper half-plane, and denote by $\Gamma_+ := \partial\mathbb{R}^2_+ = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 = 0\}$ its (infinite) boundary. Define $\Gamma_\infty := \{(x_1, x_2) \in \Gamma_+ \,:\, |x_1| > 1\}$ and let Γ_p be any curve contained in $\mathbb{R}^2_+$ connecting Γ_∞, in such a manner that $\Gamma = \Gamma_p \cup \Gamma_\infty$ corresponds to the boundary of a locally perturbed half-plane, denoted by $\Omega \subset \mathbb{R}^2_+$. Finally, denote by $\mathbf{n}$ the outer normal to Ω (see Fig. 1).

We look for time-harmonic solutions of the scalar acoustic wave equation, i.e. solutions of the form:

$$v(\mathbf{x}, t) = \mathrm{Re}\left(u_T(\mathbf{x})\, e^{-i\omega t}\right), \tag{2}$$

where u_T satisfies:

$$\Delta u_T + \kappa^2 u_T = 0, \text{ in } \Omega, \tag{3}$$

with $\kappa = \omega/c$ the wave number (where c denotes the sound speed), together with the impedance boundary condition:

$$\gamma_z(u_T) = 0, \text{ on } \Gamma, \tag{4}$$

where $\gamma_z(\cdot)$ denotes the impedance trace operator:

$$\gamma_z(u) := -\partial_n u + z\, u, \text{ on } \Gamma, \tag{5}$$

with ∂_n denoting the partial derivative in the exterior normal direction over Γ, and $z > 0$ being the impedance constant.

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