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Existence and uniqueness of a transient state for the coupled radiative–conductive heat transfer problem

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ABSTRACT

This paper deals with existence and uniqueness results for a transient nonlinear radiative–conductive system in three-dimensional case. This system describes the heat transfer for a grey, semi-transparent and non-scattering medium with general boundary conditions. We reformulate the full transient state system as a fixed-point problem. The existence and uniqueness proof is based on Banach fixed point theorem.

0. Introduction

The aim of this work is to prove the existence and uniqueness of the solution for a transient combined radiative–conductive system in three-dimensional case with general boundary conditions when the initial condition is assumed to be nonnegative. The medium is assumed grey, semi-transparent and non-scattering.

Let us consider a convex domain $\Omega \subset \mathbb{R}^3$, with C^2 boundary. Let $\mathcal{S}^2 = \{\boldsymbol{\beta} \in \mathbb{R}^3, \quad |\boldsymbol{\beta}| = 1\}$ be the unit sphere in $\mathbb{R}^3$ (the sphere of directions). Let $t \in (0, \boldsymbol{\tau})$ for $\boldsymbol{\tau} > 0$, $\boldsymbol{\mathcal{X}} = \Omega \times \mathcal{S}^2$, $\mathbf{Q}_{\tau} = (0, \boldsymbol{\tau}) \times \Omega$ and $\Sigma_{\tau} = (0, \boldsymbol{\tau}) \times \partial\Omega$. Let $\mathbf{n}$ be the outward unit normal to the boundary $\partial\Omega$. We denote

$$\partial\Omega_- = \{(x, \boldsymbol{\beta}) \in \partial\Omega \times \mathcal{S}^2 \ \text{ such that } \ \boldsymbol{\beta}.\mathbf{n} < 0\}.$$

The full system of a combined nonlinear radiation–conduction heat transfer is written in dimensionless form,

$$I(t, \boldsymbol{x}, \boldsymbol{\beta}) + \boldsymbol{\beta}.\nabla_{\boldsymbol{x}} I(t, \boldsymbol{x}, \boldsymbol{\beta}) = T^4(t, \boldsymbol{x}) \qquad (t, \boldsymbol{x}, \boldsymbol{\beta}) \in (0, \boldsymbol{\tau}) \times \boldsymbol{\mathcal{X}} \tag{1}$$

$$\partial_t T(t, \boldsymbol{x}) - \Delta T(t, \boldsymbol{x}) + 4\pi\theta T^4(t, \boldsymbol{x}) = \theta G(t, \boldsymbol{x}) \qquad (t, \boldsymbol{x}) \in \mathbf{Q}_{\tau} \tag{2}$$

$$a\partial_n T(t, \boldsymbol{x}) + bT(t, \boldsymbol{x}) = g(t, \boldsymbol{x}) \qquad (t, \boldsymbol{x}) \in \Sigma_{\tau} \tag{3}$$

$$I(t, \boldsymbol{x}, \boldsymbol{\beta}) = I_b(t, \boldsymbol{x}, \boldsymbol{\beta}) \qquad (t, \boldsymbol{x}, \boldsymbol{\beta}) \in (0, \boldsymbol{\tau}) \times \partial\Omega_- \tag{4}$$

$$T(0, \boldsymbol{x}) = T_0(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Omega \tag{5}$$

where I is the dimensionless radiation intensity, T is the dimensionless temperature, θ is a positive dimensionless constant and a and b are real numbers satisfying the condition of $|a| + |b| > 0$. The incident radiation intensity G is given by

$$G(t, \boldsymbol{x}) = \int_{\mathcal{S}^2} I(t, \boldsymbol{x}, \boldsymbol{\beta}) d\boldsymbol{\beta} \qquad (t, \boldsymbol{x}) \in \mathbf{Q}_{\tau}. \tag{6}$$

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In this paper, we assume that the mean radiation intensity of the grey medium verifies the Stefan–Boltzmann law, which is proportional to T^4. The radiative transfer equation (RTE) (1) and the conductive equation (CE) (2) are coupled via the source term $\theta\{G - 4\pi T^4\}$. We use nonhomogeneous Dirichlet boundary I_b conditions for radiation equation and different cases of boundary conditions g for CE. For a fuller treatment of the dimensionless form of radiative conductive heat transfer system, we refer the reader to [1].

Radiative–conductive heat transfer problems are the subject of various fields of engineering and science, e.g., glass manufacturing when a hot melt of glass is cooled down to room temperature. Nowadays there is a huge literature on mathematical theory in the radiative–conductive heat transfer problem, see [2–11]. For example, the paper [2] is devoted to the study of a nonstationary, nonlinear, nonlocal initial boundary value problem governing radiative–conductive heat transfer in opaque bodies with surfaces whose properties depend on the radiation frequency. This paper is a natural extension of the work done by [4], where the corresponding stationary problem was treated. In [12], the authors considered the radiative–conductive heat transfer in a scattering and absorbing medium bounded by two reflecting and radiating plane surfaces. The existence and uniqueness of a solution to this problem are established by using an iterative procedure.

In [10], M. Laitinen and T. Tiihonen studied the well-posedness of a class of models describing heat transfer by conduction and radiation in the stationary case. This theory covers different types of grey materials: semitransparent and opaque bodies as well as isotropic or non-isotropic scattering/ reflection. They also revealed that the material properties do not depend on the wavelength of the radiation.

In this paper, we consider the coupled system of nonlinear partial differential equations in three-dimensional case. In previous studies, we found theoretical of existence and uniqueness in one-dimensional case. Indeed, in the Kelley's paper [13], the authors considered a steady-state combined radiative–conductive heat transfer. In Asllanaj et al. [14] the authors generalized the Kelley's study and they proved the existence and uniqueness of the 1-D system of coupled radiative–conductive in the steady state associated to the nonhomogeneous Dirichlet boundary with the black surfaces. The medium is assumed to be a non-grey anisotropic absorbing, emitting, scattering, with axial symmetry and nonhomogeneous. They considered a nonlinear conduction equation due to the temperature dependence of the thermal conductivity. However, the approach developed by Asllanaj et al. [14] is just adaptable to 1D dimensional geometry.

We can also find in the literature some results in multidimensional case, see [15–21]. For example in [18,19] the authors considered three-dimensional stationary case. Recently, A.A. Amosov [21] proves a result on the unique solvability of a nonstationary problem of radiative–conductive heat transfer in a system of semitransparent bodies (3D case) for the homogeneous conductive boundary conditions. The radiation transfer equation is associated with boundary conditions of mirror reflection and refraction according to the Fresnel laws is used to describe the propagation of radiation.

Moreover, M.M. Porzio and Ó. López Pouso proved in [22] an existence and uniqueness theorem for the non-grey coupled convection–conduction–radiation system associated to the mixed nonhomogeneous Dirichlet and homogeneous Neumann boundary conditions by means of accretive operators theory. Leaving aside the grey or non-grey character, the main difference between our problem and the one studied in [22] is that we do not include the transient term in the RTE. This is an interesting point because this term is really negligible in a wide range of applications; e.g., thermoforming glass see [1,23] and references therein. Moreover, the techniques used in [22] do not allow disregarding it. In our study, we also discuss different types of boundary conditions.

In this paper, we prove the existence and uniqueness of solution for the nonlinear radiative–conductive system in 3-dimensional case associated to the nonhomogeneous Dirichlet boundary conditions for radiation equation and for different types of conductive boundary conditions. The Banach fixed point theorem is the principal tool used to solve this problem.

Recently, some attention has been accorded to numerical methods to study the radiative transfer and the nonlinear radiative–conductive heat transfer problem including optimal control problems, for more details see [1,14,24–37] and the pioneering book [38] and references therein. Asllanaj et al. [23] simulated transient heat transfer by radiation and conduction in two-dimensional complex shaped domains with structured and unstructured triangular meshes working with an absorbing, emitting and non-scattering grey medium.

The plan of this paper is as follows: Section 2, contains the statement of the main result (Theorem 1.1). Section 2.3 is devoted to its proof based on Banach fixed point theorem.

1. Main results

In order to state the main result, we introduce the following notations

$$L^p(\mathbf{Q}_\tau) := L^p(0, \boldsymbol{\tau}; L^p(\Omega)) \text{ for all } p \in [1, \infty[,$$

$$W_p^{2,1}(\mathbf{Q}_\tau) := \{\phi \text{ such that } \phi, \phi_t, \phi_{x_i}, \phi_{x_i,x_j} \in L^p(\mathbf{Q}_\tau)\} \; \forall \; p \in [1, \infty[.$$

According to the method introduced in [39–41] to solve the neutron equations, we consider the following space

$$\mathcal{W}^2 = \{v \in L^2(\boldsymbol{\mathcal{X}}) \text{ such that } \boldsymbol{\beta}.\nabla_x v \in L^2(\boldsymbol{\mathcal{X}})\}$$

and the following subset of $\partial\Omega \times \boldsymbol{\mathcal{S}}^2$

$$\partial\Omega_+ = \{(x, \boldsymbol{\beta}) \in \partial\Omega \times \boldsymbol{\mathcal{S}}^2 \text{ and } \boldsymbol{\beta}.\mathbf{n} > 0\}.$$